# Extensions of dualities and a new approach to de Vries' duality

G. Dimov
Joint work with E. Ivanova-Dimova and W. Tholen

Faculty of Math. and Informatics, University of Sofia, Bulgaria

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# De Vries' Duality

The celebrated Stone Duality Theorem shows that the entire information about a zero-dimensional compact Hausdorff space (= *Stone space*) $X$ is, up to homeomorphism, contained in its Boolean algebra $(\mathrm{CO}(X), \subseteq)$ of all clopen (= closed and open) subsets of $X$. Likewise, all information about the continuous maps between two such spaces $X$ and $Y$ is encoded by the Boolean homomorphisms between the Boolean algebras $(\mathrm{CO}(Y), \subseteq)$ and $(\mathrm{CO}(X), \subseteq)$. It is natural to ask whether a similar result holds for all compact Hausdorff spaces and continuous maps between them.

The first candidate for the role of the Boolean algebra CO($X$) under such an extension seems to be the Boolean algebra (RC($X$), $\subseteq$) of all regular closed subsets of a compact Hausdorff space $X$ (denoted briefly by $RC(X)$), but it fails immediately since, as is well-known, RC($X$) is isomorphic to RC($EX$), where $EX$ is the absolute of $X$. However, in 1962, de Vries showed that, if we regard the Boolean algebra RC($X$) together with the relation $\ll_X$ on RC($X$), defined by

$$F \ll_X G \Leftrightarrow F \subseteq \mathrm{int}_X(G).$$

then the pair (RC($X$), $\ll_X$) determines uniquely (up to homeomorphism) the compact Hausdorff space $X$.

Moreover, with the help of some special maps between $(\mathrm{RC}(X), \ll_X)$ and $(\mathrm{RC}(Y), \ll_Y)$, where $X$ and $Y$ are compact Hausdorff spaces, one can reconstruct all continuous maps between $Y$ and $X$. De Vries gave an algebraic description of the pairs $(\mathrm{RC}(X), \ll_X)$ as pairs $(A, \ll)$, formed by a complete Boolean algebra $A$ and a relation $\ll$ on $A$, satisfying the following axioms:

- ($\ll$1) $a \ll b$ implies $a \leq b$.
- ($\ll$2) $0 \ll 0$.
- ($\ll$3) $a \leq b \ll c \leq t$ implies $a \ll t$.
- ($\ll$4) $a \ll c$ and $b \ll c$ implies $a \vee b \ll c$.
- ($\ll$5) If $a \ll c$ then $a \ll b \ll c$ for some $b \in B$.
- ($\ll$6) If $a \neq 0$ then there exists $b \neq 0$ such that $b \ll a$.
- ($\ll$7) $a \ll b$ implies $b^* \ll a^*$.

These abstract pairs $(A, \ll)$ are now called *de Vries' algebras*.

De Vries also described algebraically the needed special maps $\varphi$ between such pairs $(A, \ll)$ (they are now called *de Vries morphisms*); they satisfy the following axioms:

(DV1) $\varphi(0) = 0$;
(DV2) $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$, for all $a, b \in A$;
(DV3) If $a, b \in A$ and $a \ll b$, then $(\varphi(a^*))^* \ll \varphi(b)$;
(DV4) $\varphi(a) = \bigvee\{\varphi(b) \mid b \ll a\}$, for every $a \in A$.

The composition "$\diamond$" of two such maps $\varphi_1 : (A_1, \ll_1) \longrightarrow (A_2, \ll_2)$ and $\varphi_2 : (A_2, \ll_2) \longrightarrow (A_3, \ll_3)$ is defined by

$$(\varphi_2 \diamond \varphi_1)(a) \stackrel{\text{df}}{=} \bigvee\{(\varphi_2 \circ \varphi_1)(b) \mid b \ll_1 a\}.$$

In this way de Vries obtained a category **DeV** and proved that it is dually equivalent with the category **CHaus** of compact Hausdorff spaces and continuous maps. It is easy to see that:

**Fact 1.** (de Vries) If $\varphi : (A, \ll) \longrightarrow (A', \ll')$ is a de Vries morphism, then:
(a) $\varphi(1_A) = 1_{A'}$;
*(b)* for every $a \in A$, $\varphi(a^*) \leq (\varphi(a))^*$.

De Vries noticed that his duality extends the restriction of the Stone Duality to the category **CBoole** of complete Boolean algebras and Boolean homomorphisms (i.e., the duality

$$T \stackrel{\text{df}}{=} S^a \upharpoonright_{\mathbf{CBoole}} : \mathbf{CBoole} \longrightarrow \mathbf{EDCHaus}$$

where **EDCHaus** is the category of extremally disconnected compact Hausdorff spaces and continuous maps).

The main goal of this talk is to present a general categorical theorem for extension of dualities and to obtain with its help a completely new proof of de Vries' Duality extending the above duality $T$. In the process of doing this, we will also construct a new category **StoneDeV**, isomorphic to the category **DeV**, such that

$$|\mathbf{StoneDeV}| = |\mathbf{DeV}|$$

but its morphisms are:

(1) sets of *Boolean homomorphisms* preserving the relation $\ll$, and

(2) *their composition is a natural one*.

As well, the recent Bezhanishvili-Morandi-Olberding Duality Theorem which extends the de Vries duality to the category **Tych** of Tychonoff spaces and continuous maps will be derived from our general Extension Theorem for Dualities.

# Extensions of dualities

As it was explained in the talk of Tholen, for obtaining an extension of the above dual equivalence $T$ to a dual equivalence $\tilde{T} : \mathcal{D} \longrightarrow$ **CHaus**, we cannot use the Extension Theorem for Dualities presented there because the concrete class $\mathcal{P}$ of all irreducible **CHaus**-morphisms whose domain is an **EDCHaus**-object does not satisfy the general condition (P5) of that theorem. We will prove, however, a new Extension Theorem for Dualities and will obtain, with its help, such an extension $\tilde{T}$.

Let us recall *the general problem*: given a dual equivalence $T : \mathcal{A} \longrightarrow \mathcal{B}$ and an embedding $I$ of $\mathcal{B}$ as a full subcategory of a category $\mathcal{C}$, find a *natural construction* for a category $\mathcal{D}$ into which $\mathcal{A}$ may be fully embedded via $J$, such that $T$ extends to a dual equivalence $\tilde{T} : \mathcal{D} \longrightarrow \mathcal{C}$.

Our construction depends on a class $\mathcal{P}$ of morphisms of $\mathcal{C}$ satisfying certain conditions, which are closely related to certain properties of the full embedding $I$.

We call a class $\mathcal{P}$ of morphisms in $\mathcal{C}$ a *weak* $(\mathcal{B},\mathcal{C})$*-covering class* if it satisfies the following conditions:

(P1) $\forall\ (p\colon B \longrightarrow C)\ \in \mathcal{P}:\ B \in |\mathcal{B}|$;

(P2) $\forall B \in |\mathcal{B}| : 1_B \in \mathcal{P}$;

(P3) $\mathcal{P} \circ \mathrm{Iso}(\mathcal{B}) \subseteq \mathcal{P}$;

(P4) $\forall C \in |\mathcal{C}|\ \exists\ (p : B \longrightarrow C) \in \mathcal{P}$;

(P5°) *for morphisms in* $\mathcal{C}$*, there is an assignment*
$((p\colon B \to C) \in \mathcal{P}, v\colon C \to C', (p'\colon B' \to C') \in \mathcal{P}) \mapsto$
$(\hat{v}\colon B \to B'$ *with* $v \circ p = p' \circ \hat{v})$.

Note that in the given assignment, $\hat{v}$ depends not only on $v$, but also on $p$ and $p'$.

Later on we will see that, when $\mathcal{B}$ is projective in $\mathcal{C}$, such a class $\mathcal{P}$ always exists.

As a precursor to the category $\mathcal{D}$, we consider the comma category $(IT \downarrow_{\mathcal{P}} \mathcal{C})$, defined as follows:

- objects in $(IT \downarrow_{\mathcal{P}} \mathcal{C})$ are pairs $(A, p)$ with $A \in |\mathcal{A}|$ and $p : TA \longrightarrow C$ in the class $\mathcal{P}$;
- morphisms $(\varphi, f) : (A, p) \longrightarrow (A', p')$ in $(IT \downarrow_{\mathcal{P}} \mathcal{C})$ are given by morphisms $\varphi : A \longrightarrow A'$ in $\mathcal{A}$ and $f : C' \longrightarrow C$ in $\mathcal{C}$, such that $p \circ T\varphi = f \circ p'$;
- composition is as in $\mathcal{A}$ and $\mathcal{C}$; that is, $(\varphi, f)$ as above gets composed with $(\varphi', f') : (A', p') \longrightarrow (A'', p'')$ by the horizontal pasting of diagrams, that is,

$$(\varphi', f') \circ (\varphi, f) \stackrel{\text{df}}{=} (\varphi' \circ \varphi, f \circ f').$$

- the identity morphism of a $(IT \downarrow_{\mathcal{P}} \mathcal{C})$-object $(A, p)$ is the $(IT \downarrow_{\mathcal{P}} \mathcal{C})$-morphism $(1_A, 1_{\text{cod}(p)})$.

On the hom-sets of $(IT \downarrow_{\mathcal{P}} \mathcal{C})$ we define a compatible equivalence relation by

$$(\varphi, f) \sim (\psi, g) \iff f = g,$$

for all $(\varphi, f), (\psi, g) : (A, p) \longrightarrow (A', p')$. We denote the equivalence class of $(\varphi, f)$ by $[\varphi, f]$ (or $[\varphi, f]_{(A,p),(A',p')}$, if clarity demands it), and let $\mathcal{D}$ be the quotient category

$$(IT \downarrow_{\mathcal{P}} \mathcal{C})/ \sim .$$

Thanks to (P2), we have the functor $J : \mathcal{A} \longrightarrow \mathcal{D}$, defined by

$$(\varphi : A \longrightarrow A') \mapsto (\ J\varphi \stackrel{\text{df}}{=} [\varphi, T\varphi] : (A, 1_{TA}) \longrightarrow (A', 1_{TA'})\ ),$$

which is easily seen to be a full embedding.

Given a dual equivalence $(S, T, \eta, \varepsilon)$ with contravariant functors

$$T : \mathcal{A} \longrightarrow \mathcal{B} \quad \text{and} \quad S : \mathcal{B} \longrightarrow \mathcal{A}$$

and natural isomorphisms $\eta : \mathrm{Id}_{\mathcal{B}} \longrightarrow T \circ S$ and $\varepsilon : \mathrm{Id}_{\mathcal{A}} \longrightarrow S \circ T$, it is now straightforward to establish a dual equivalence of $\mathcal{D}$ with $\mathcal{C}$, as follows:

**Theorem 1.** *There is a dual equivalence $\tilde{T} : \mathcal{D} \longleftrightarrow \mathcal{C} : \tilde{S}$ extending the given dual equivalence $T : \mathcal{A} \longleftrightarrow \mathcal{B} : S$, in the sense that that $\tilde{T}J = IT$ and $\tilde{S}I \cong JS$.*

*The unit $\tilde{\eta} : \mathrm{Id}_{\mathcal{C}} \longrightarrow \tilde{T}\tilde{S}$ and the counit $\tilde{\varepsilon} : \mathrm{Id}_{\mathcal{D}} \longrightarrow \tilde{S}\tilde{T}$ of the extended adjunction and the natural isomorphism $\gamma : JS \longrightarrow \tilde{S}I$ may be chosen to satisfy the identities $\tilde{\eta} = 1_{\mathrm{Id}_{\mathcal{C}}}$, $\tilde{T}\tilde{\varepsilon} = 1_{\tilde{T}}$, $\tilde{\varepsilon}\tilde{S} = 1_{\tilde{S}}$, and $\tilde{T}\gamma = I\eta$, $\gamma T \circ J\varepsilon = \tilde{\varepsilon}J$.*

# The construction of $\tilde{T}$ and $\tilde{S}$

The contravariant functor $\tilde{T} : \mathcal{D} \longrightarrow \mathcal{C}$ is given by

$$\tilde{T}(A, p) \stackrel{\text{df}}{=} \mathrm{cod}(p) \text{ and } \tilde{T}([\varphi, f]) \stackrel{\text{df}}{=} f.$$

To define the contravariant functor $\tilde{S}$ on objects, one chooses for every $C \in |\mathcal{C}|$ a morphism $\pi_C : EC \longrightarrow C$ in $\mathcal{P}$, with $\pi_B = 1_B$ for all $B \in |\mathcal{B}|$ (according to (P2)), and then puts

$$\tilde{S}C \stackrel{\text{df}}{=} (SEC, \pi_C \circ \eta_{EC}^{-1}).$$

# The construction of $\tilde{T}$ and $\tilde{S}$

For a morphism $f : C' \longrightarrow C$ in $\mathcal{C}$, (P5°) and the fullness of $T$ allow one to choose a morphism $\varphi_f : SEC \longrightarrow SEC'$ in $\mathcal{A}$ with $\pi_C \circ \eta_{EC}^{-1} \circ T\varphi_f = f \circ \pi_{C'} \circ \eta_{EC'}^{-1}$; then we put

$$\tilde{S}f \stackrel{\text{df}}{=} [\varphi_f, f].$$

It is easy to see that:

**Proposition 1.** $\mathcal{B}$ is a coreflective subcategory of $\mathcal{C}$ if, and only if, there exists a class $\mathcal{P}$ of $\mathcal{C}$-morphisms satisfying properties (P1-4) and the following strengthening of (P5°):

(P5*) *for all* $v : C \longrightarrow C'$ *in* $\mathcal{C}$ *and* $p : B \longrightarrow C$, $p' : B' \longrightarrow C'$ *in* $\mathcal{P}$, *there is precisely one morphism* $\hat{v} : B \longrightarrow B'$ *with* $v \circ p = p' \circ \hat{v}$.

If the class $\mathcal{P}$ satisfies conditions (P1-4) and (P5*), then the equivalence relation $\sim$ is just the equality relation. Thus, in this case, the category $\mathcal{D}$ coincides with the category $(IT \downarrow_{\mathcal{P}} \mathcal{C})$. In the sequel, we will also use the dualization of this special form of Theorem 1.

Recall that, for a class $\mathcal{Q}$ of morphisms in $\mathcal{C}$, an object $B \in |\mathcal{C}|$ is $\mathcal{Q}$-*projective* if, for all $(q : C \longrightarrow D) \in \mathcal{Q}$, the map

$$\mathcal{C}(B, q) : \mathcal{C}(B, C) \longrightarrow \mathcal{C}(B, D), \quad h \mapsto q \circ h,$$

is surjective. Since this map is trivially bijective when $q$ is an isomorphism, without loss of generality we may assume that $\mathcal{Q}$ contain all isomorphisms and be closed under composition with them. We call a full subcategory $\mathcal{B}$ in $\mathcal{C}$ *projective* if there is a such a class $\mathcal{Q}$ satisfying

(Q1) $\forall C \in |\mathcal{C}| \ \exists\, (q : B \longrightarrow C) \in \mathcal{Q}$ with $B \in |\mathcal{B}|$;

(Q2) $\forall B \in |\mathcal{B}| : \quad B$ is $\mathcal{Q}$-projective.

**Proposition 1.** *A full subcategory $\mathcal{B}$ of a category $\mathcal{C}$ is projective if, and only if, there is a weak $(\mathcal{B}, \mathcal{C})$-covering class $\mathcal{P}$.*

# A new approach to the de Vries duality

In view of the previous section, *throughout this section we use the following notation:*

$$\mathcal{A} \stackrel{\text{df}}{=} \textbf{CBoole}, \quad \mathcal{B} \stackrel{\text{df}}{=} \textbf{EDCHaus}, \quad \mathcal{C} \stackrel{\text{df}}{=} \textbf{CHaus},$$

*with $I : \mathcal{B} \hookrightarrow \mathcal{C}$ denoting the inclusion functor;*

*$\mathcal{P}$ denotes the class of all irreducible continuous maps between compact Hausdorff spaces with domain in $|\mathcal{B}|$.*

Trivially, $\mathcal{B}$ is a full subcategory of $\mathcal{C}$ that is closed under $\mathcal{C}$-isomorphisms. By the results of Gleason, the class $\mathcal{P}$ satisfies conditions (P1-4), (P5°) of the previous Section (and $\mathcal{B}$ is a projective subcategory of $\mathcal{C}$).

With the restrictions

$$T \stackrel{\mathrm{df}}{=} S^a \restriction_{\mathcal{A}} \quad \text{and} \quad S \stackrel{\mathrm{df}}{=} S^t \restriction_{\mathcal{B}}$$

of the functors furnishing the Stone Duality, using the well-known Stone's result, we obtain the contravariant functors $T : \mathcal{A} \longrightarrow \mathcal{B}$ and $S : \mathcal{B} \longrightarrow \mathcal{A}$. They realize a dual equivalence between the categories $\mathcal{A}$ and $\mathcal{B}$.

Defining the category $\mathcal{D}$ as in Theorem 1, we obtain the full embedding $J : \mathcal{A} \longrightarrow \mathcal{D}$ and the dual equivalence $\tilde{T} : \mathcal{D} \longrightarrow \mathcal{C}$ which extends the dual equivalence $T : \mathcal{A} \longrightarrow \mathcal{B}$, so that $I \circ T = \tilde{T} \circ J$, as given by Theorem 1. We now prove that the categories **DeV** and $\mathcal{D}$ are equivalent, thus completing our alternative proof of de Vries Duality Theorem. This will be done in several steps. In one of them, we will obtain a new category dual to the category **CHaus**.

**Definition.**

Let $(A, \ll), (A', \ll')$ be two de Vries' algebras. Then a Boolean homomorphism $\varphi : A \longrightarrow A'$ will be called a *Fedorchuk homomorphism* (briefly, *F-homomorphism*) if $a \ll b$ implies $\varphi(a) \ll' \varphi(b)$, for all $a, b \in A$.

**Definition.**

Let $(A, \ll)$ be a de Vries algebra, $B$ be a complete Boolean algebra and $\varphi : A \longrightarrow B$ be a function. Then the function $V(\varphi) : A \longrightarrow B$, defined by

$$(V(\varphi))(a) \stackrel{\text{df}}{=} \bigvee \{\varphi(b) \mid b \ll a\},$$

for every $a \in A$, will be called a *de Vries transformation of the function* $\varphi$.

# The definition of the category StoneDeV.

We set

$$|\textbf{StoneDeV}| \stackrel{\text{df}}{=} |\textbf{DeV}|.$$

Further, for every $(A, \ll), (A', \ll') \in |\textbf{StoneDeV}|$, we define

$$\textbf{StoneDeV}((A, \ll), (A', \ll')) \stackrel{\text{df}}{=}$$

$$\{\langle\varphi\rangle \mid \varphi : (A, \ll) \longrightarrow (A', \ll') \text{ is an F-homomorphism}\},$$

where $\langle\varphi\rangle$ is the equivalence class of $\varphi$ under the equivalence relation $\simeq$ in the set of all Fedorchuk homomorphisms between $(A, \ll)$ and $(A', \ll')$ defined by

$$\varphi \simeq \psi \Leftrightarrow V(\varphi) = V(\psi).$$

# The definition of the category StoneDeV.

The **StoneDeV**-composition between two **StoneDeV**-morphisms $\langle\varphi\rangle : (A, \ll) \longrightarrow (A', \ll')$ and $\langle\psi\rangle : (A', \ll') \longrightarrow (A'', \ll'')$ is defined as follows:

$$\langle\psi\rangle \circ \langle\varphi\rangle \stackrel{\text{df}}{=} \langle\psi \circ \varphi\rangle.$$

Finally, for every **StoneDeV**-object $(A, \ll)$, its **StoneDeV**-identity is

$$1_{(A,\ll)} \stackrel{\text{df}}{=} \langle 1_A \rangle.$$

**Theorem 2.**

The categories **CHaus** and **StoneDeV** are dually equivalent.

**Lemma 1.**

Let $(A, \ll)$ and $(A', \ll')$ be two complete normal contact algebras and $\varphi : (A, \ll) \longrightarrow (A', \ll')$ be a Fedorchuk homomorphism. Then $V(\varphi)$ is a de Vries morphism.

**Theorem 3.**

The categories **StoneDeV** and **DeV** are isomorphic.

**Corollary. (de Vries)**

The categories **CHaus** and **DeV** are dually equivalent.

# A new approach to the Bezhanishvili-Morandi-Olberding duality

Recently, G. Bezhanishvili, P.J. Morandi and B. Olberding described a category **BMO** and a dual equivalence of **BMO** with the category **Tych** of Tychonoff spaces and continuous maps which extends de Vries' dual equivalence $\Psi^a : \mathbf{DeV} \longrightarrow \mathbf{CHaus}$. In this section we will derive the Bezhanishvili-Morandi-Olberding Duality Theorem from the dualization of the very particular case of our Theorem 1 when the class $\mathcal{P}$ satisfies the axioms (P1-4) and (P5$^*$). We will first formulate this dualization explicitly.

Let $\mathcal{A}$ be a full subcategory of a category $\mathcal{D}$ with inclusion functor $J$. We call a class $\mathcal{J}$ of morphisms in $\mathcal{D}$ a *strong* $(\mathcal{A}, \mathcal{D})$-*insertion class* if it satisfies the following conditions (J1-4) and (J5$^*$):

(J1) $\forall\ (j : D \longrightarrow A)\ \in \mathcal{J} :\ A \in |\mathcal{A}|$;

(J2) $\forall A \in |\mathcal{A}| : 1_A \in \mathcal{J}$;

(J3) $\mathrm{Iso}(\mathcal{A}) \circ \mathcal{J} \subseteq \mathcal{J}$;

(J4) $\forall D \in |\mathcal{D}|\ \exists\ (j : D \longrightarrow A) \in \mathcal{J}$;

(J5$^*$) *for all* $v : D \longrightarrow D'$ *in* $\mathcal{D}$ *and* $j : D \longrightarrow A,\ j' : D' \longrightarrow A'$ *in* $\mathcal{J}$, *there is precisely one morphism* $\overline{v} : A \longrightarrow A'$ *with* $j' \circ v = \overline{v} \circ j$.

Again, we point out that, in the given assignment, $\overline{v}$ depends not only on $v$, but also on $j$ and $j'$. Next, we note that, in the presence of (J3), condition (J2) means equivalently

(J2′) $\mathrm{Iso}(\mathcal{A}) \subseteq \mathcal{J}$.

In condition (J4) we tacitly assume that, for every $D \in |\mathcal{D}|$, we have a *chosen* morphism $j \in \mathcal{J}$ with domain $D$. In the presence of (J2), that morphism may be taken to be an identity morphism whenever $D \in |\mathcal{A}|$. To emphasize the choice, we may reformulate (J4), as follows:

(J4′) $\forall D \in |\mathcal{D}|\ \exists\, (\rho_D : D \longrightarrow FD) \in \mathcal{J}$ *(with* $\rho_D = 1_D$ *when* $D \in |\mathcal{A}|$*).*

It is now clear that (J5$^*$) enables us to make $F$ a functor $\mathcal{D} \longrightarrow \mathcal{A}$ and $\rho$ a natural transformation $\mathrm{Id}_{\mathcal{C}} \longrightarrow JF$. Dualizing Proposition 1, we obtain the following assertion:

**Proposition 2.** *The full subcategory $\mathcal{A}$ of $\mathcal{D}$ is reflective in $\mathcal{D}$ if, and only if, there is a strong $(\mathcal{A}, \mathcal{D})$-insertion class $\mathcal{J}$ of morphisms in $\mathcal{D}$.*

In addition to the full subcategory $\mathcal{A}$ of $\mathcal{D}$ with inclusion functor $J$ and a strong $(\mathcal{A},\mathcal{D})$-insertion class $\mathcal{J}$ we consider again a dual equivalence $(S, T, \eta, \varepsilon)$ with contravariant functors

$$T:\mathcal{A}\longrightarrow\mathcal{B} \quad \text{and} \quad S:\mathcal{B}\longrightarrow\mathcal{A}$$

and natural isomorphisms
$\eta:\mathrm{Id}_{\mathcal{B}}\longrightarrow T\circ S$ and $\varepsilon:\mathrm{Id}_{\mathcal{A}}\longrightarrow S\circ T$.

We then construct the category $\mathcal{C}$, as follows:

- objects in $\mathcal{C}$ are pairs $(B, j)$ with $B \in |\mathcal{B}|$ and $j : D \longrightarrow SB$ in the class $\mathcal{J}$;
- morphisms $(\varphi, f) : (B, j) \longrightarrow (B', j')$ in $\mathcal{C}$ are given by morphisms $\varphi : B \longrightarrow B'$ in $\mathcal{B}$ and $f : D' \longrightarrow D$ in $\mathcal{D}$, such that, in the notation of (J5$^*$), $S\varphi = \overline{f}$;
- composition is as in $\mathcal{B}$ and $\mathcal{D}$; that is, $(\varphi, f)$ as above gets composed with $(\varphi', f') : (B', j') \longrightarrow (B'', j'')$ by the horizontal pasting of diagrams, that is,

$$(\varphi', f') \circ (\varphi, f) \stackrel{\text{df}}{=} (\varphi' \circ \varphi, f \circ f').$$

- the identity morphism of a $\mathcal{C}$-object $(B, j)$ is the $\mathcal{C}$-morphism $(1_B, 1_{\mathrm{dom}(j)})$.

With (J2) one obtains the full embedding $I : \mathcal{B} \longrightarrow \mathcal{C}$, defined by

$$(\varphi : B \longrightarrow B') \mapsto (\ I\varphi \stackrel{\text{df}}{=} (\varphi, S\varphi) : (B, 1_{SB}) \longrightarrow (B', 1_{SB'})\ ).$$

A dual equivalence

$$\overline{S} : \mathcal{C} \longleftrightarrow \mathcal{D} : \overline{T}$$

may now be established, as follows:

- $\overline{S}\ :\ ((\varphi, f) : (B, j) \longrightarrow (B', j))\ \mapsto\ (f : \mathrm{dom}(j') \longrightarrow \mathrm{dom}(j));$
- $\overline{T}\ :\ (f : D' \longrightarrow D)\ \mapsto\ ((\varphi_f, f) : (TFD, \varepsilon_{FD} \circ \rho_D) \longrightarrow (TFD', \varepsilon_{FD'} \circ \rho_{D'})),$
  where $\varphi_f : TFD \longrightarrow TFD'$ is the unique $\mathcal{B}$-morphism such that $S\varphi_f = \overline{f}(j', j)$ with $j = \varepsilon_{FD} \circ \rho_D$ and $j' = \varepsilon_{FD'} \circ \rho_{D'}$.

The dualization of the particular case of Theorem 1 now reads as follows:

**Theorem 4.** $(\overline{T}, \overline{S}, \overline{\varepsilon}, \overline{\eta})$ *is a dual equivalence with* $\overline{S}\,\overline{T} = \mathrm{Id}_{\mathcal{D}}$*, extending the given dual equivalence* $(T, S, \varepsilon, \eta)$*, so that* $\overline{S}I = JS$ *and* $\overline{T}J \cong IT$.

We note that, as $\mathcal{A}$ is reflective in $\mathcal{D}$, $\mathcal{B}$ is coreflective in $\mathcal{C}$.

We set (and *we will keep this notation throughout this section*)

$$\mathcal{A} \stackrel{\text{df}}{=} \mathbf{CHaus}, \ \ \mathcal{B} \stackrel{\text{df}}{=} \mathbf{DeV}, \ \ \mathcal{D} \stackrel{\text{df}}{=} \mathbf{Tych}, \ \ S \stackrel{\text{df}}{=} \Psi^a,$$

$$\mathcal{J} \stackrel{\text{df}}{=} \{j : X \to Y \mid X \in |\mathbf{Tych}|, Y \in |\mathbf{CHaus}|,$$

$$j \text{ is a dense embedding}, j(X) \underset{\mathrm{C}^*}{\subseteq} Y\},$$

where $j(X) \underset{\mathrm{C}^*}{\subseteq} Y$ means that $j(X)$ is $C^*$-embedded in $Y$, and we denote by

$$J : \mathcal{A} \hookrightarrow \mathcal{D}$$

the inclusion functor. Note that we regard as elements of the class $\mathcal{J}$ all representatives of the Stone-Čech compactifications of Tychonoff spaces. Obviously, the class $\mathcal{J}$ satisfies conditions (J1-4) and (J5$^*$) (and $\mathcal{A}$ is a reflective subcategory of $\mathcal{D}$).

Therefore, we can apply Theorem 4. It gives us a category $\mathcal{C}$, a dual equivalence

$$\overline{S} : \mathcal{C} \longrightarrow \mathcal{D}$$

and a full embedding $I : \mathcal{B} \longrightarrow \mathcal{C}$ such that

$$\overline{S} \circ I = J \circ S.$$

Then we prove the following theorem:

**Theorem 5.** The categories **BMO** and $\mathcal{C}$ are equivalent.

This obviously implies that:

**Theorem 5.** (BMO) There exists a dual equivalence between the categories **BMO** and **Tych** which extend the de Vries dual equivalence between the categories **DeV** and **CHaus**.

# Thank You!