A Generalization of Gelfand-Naimark-Stone Duality to Completely Regular Spaces

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Our goal is to extend the duality from compact Hausdorff spaces to completely regular spaces.

From Spaces to Algebras

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$C(X)$ is complete with respect to the **uniform norm**

$$\|f\| = \sup\{|f(x)| : x \in X\} = \inf\{r \in \mathbb{R} : |f| \leq r\}.$$

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C is a functor from **KHaus** to $\boldsymbol{uba\ell}$.

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Y is a functor from $\boldsymbol{ba\ell}$ to **KHaus**.

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Given a compactification $e : X \to Y$, we can associate $C(Y)$ to Y. What can we associate to X? If we use $C(X)$, this is not bounded. If we use $C^*(X) \cong C(\beta X)$, we only recover βX and not X.

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Our choice is to use the ℓ-algebra $B(X)$ of all bounded real-valued functions on X.

Basic Algebras

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We call $B \in \boldsymbol{ba\ell}$ a **basic algebra** if it is Dedekind complete and its boolean algebra of idempotents is atomic.

A Ring-Theoretic Version of Tarski Duality

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Theorem. The category of basic algebras is dual to the category of sets.

Compactifications

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Morphisms in **Comp** are pairs (f, g) of continuous maps such that the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\quad e \quad} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \xrightarrow[\quad e' \quad]{} & Y' \end{array}$$

Basic Extensions

Proposition. Let $e : X \to Y$ be a compactification. Then the map $e^\flat : C(Y) \to B(X)$ sending f to $f \circ e$ is a 1-1 $\boldsymbol{ba\ell}$ morphism such that each element of $B(X)$ is a join of meets from the image.

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Definition. A **basic extension** is a monomorphism $\alpha : A \to B$ in ***baℓ*** with B basic such that $\alpha[A]$ is join-meet dense in B. These form a category ***basic***.

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Morphisms in ***basic*** are pairs (ρ, σ) making the diagram commute.

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\sigma} \\ A' & \xrightarrow[\alpha']{} & B' \end{array}$$

Dual Equivalence of **Comp** and ***ubasic***

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Theorem. There is a dual adjunction between **Comp** and ***basic***, sending $e : X \to Y$ to $C(Y) \to B(X)$, and $\alpha : A \to B$ to $X_B \to Y_A$.

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ubasic is the reflective subcategory ***basic*** consisting of basic extensions $\alpha : A \to B$ with $A \in$ ***ubaℓ***.

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Theorem. The dual adjunction restricts to a dual equivalence between **Comp** and ***ubasic***.

Stone-Čech Compactifications

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This yields that there is an equivalence between **CReg** and **SComp**.

By our duality theorem, **CReg** is then dually equivalent to a full subcategory of ***basic***. This subcategory will consist of "maximal" extensions.

Compatible Basic Extensions

Suppose we have a commutative diagram of compactifications of X.

$$\begin{array}{ccc} X & \xrightarrow{e} & Y \\ & {}_{e'}\searrow & \downarrow{\scriptstyle f} \\ & & Y' \end{array}$$

The topology on X can be described as the smallest topology making e continuous, or the smallest topology making e' continuous.

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Definition. We call two basic extensions $\alpha : A \to B$ and $\gamma : C \to B$ **compatible** if both yield the same topology on X_B.

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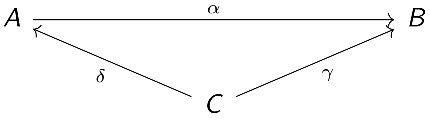
- We say that a basic extension $\alpha : A \to B$ is **maximal** provided for every compatible basic extension $\gamma : C \to B$, there is a ***baℓ*** morphism $\delta : C \to A$ such that $\alpha \circ \delta = \gamma$.



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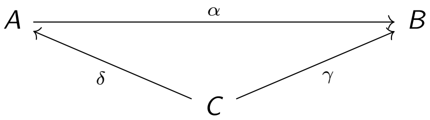

- Let ***mbasic*** be the full subcategory of ***basic*** consisting of maximal basic extensions.

Generalization of Gelfand-Naimark-Stone Duality

Theorem. For a compactification $e : X \to Y$ the following are equivalent.

- The basic extension $C(Y) \to B(X)$ is maximal.
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Theorem. There is a dual equivalence between ***mbasic*** and **SComp**, and so there is a dual equivalence between **CReg** and **SComp**.

Therefore, ***mbasic*** is a category of algebraic objects dually equivalent to the category of completely regular spaces.

We have used these results to characterize such topological properties as normality, Lindelöf, and local compactness in terms of basic extensions that are subject to additional axioms.

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Thanks to the organizers for the invitation to speak at this conference and thanks for your attention.