Axiomatising categories of spaces: the case of compact Hausdorff spaces

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Czech Academy of Sciences

What is the logic of compact Hausdorff spaces?

- Lawvere's Elementary Theory of the Category of Sets (1964)
- Hu's Primal Algebra Theorem (1969)
- Lindström's Theorem for First Order Logic (1969)

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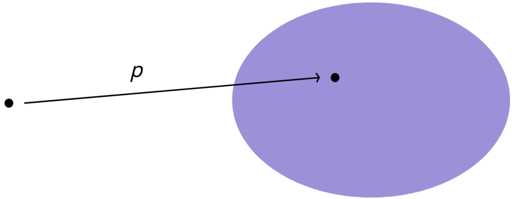
- **In this talk:** a different solution to this "axiomatisation problem" (intuitions from universal algebra, duality, etc. . .)

Given $X \in \mathsf{KH}$, there is a bijection $X \cong \hom_{\mathsf{KH}}(\mathbf{1}, X)$.



- KH has **enough points**: any two distinct maps $f, g \colon X \rightrightarrows Y$ must differ on (at least) one point.

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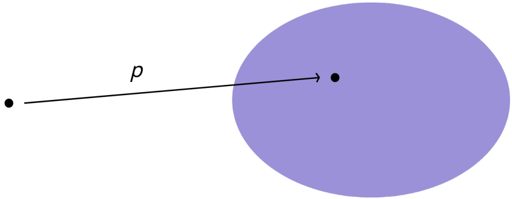

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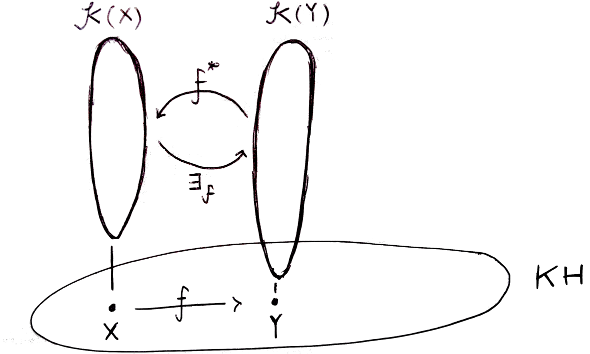
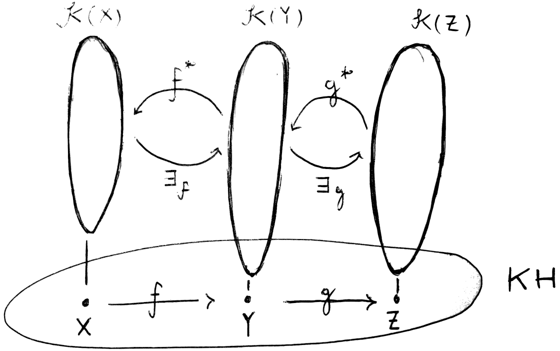
- $\mathsf{BA}^{\mathrm{op}}$ is well-pointed $\leftrightarrow$ **Maximal Ideal Theorem**



$\mathcal{K}(X) =$ lattice of closed subsets of X, $f^* = f^{-1}$, $\exists_f = f(-)$



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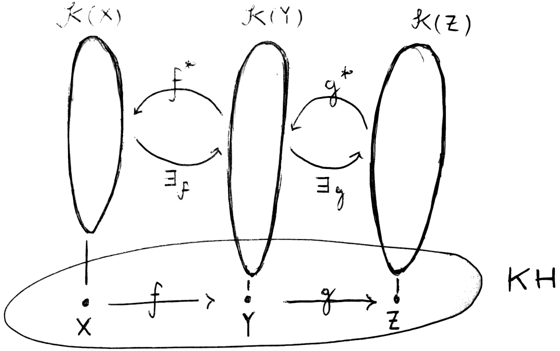

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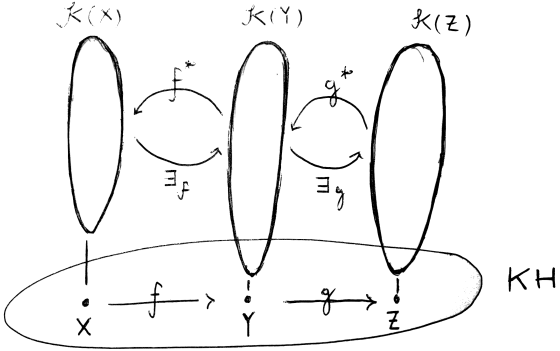

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- The functor $\mathcal{K}\colon \mathsf{KH}^{\mathrm{op}} \to \mathsf{DL}$ describes the structure of **coherent category** of KH
- KH is also **positive**: the coproduct of two spaces is disjoint

Filtrality

Let L be a distributive lattice,

- $\mathcal{F}(L)$ the lattice of non-empty **filters**, ordered by $\supseteq$,
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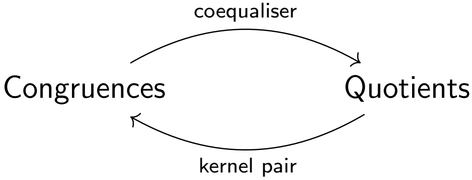
- In KH: X filtral $\Leftrightarrow$ X is a Boolean space
- Each KH space is covered by a filtral one $\to$ **filtral category**

Exactness, or "Maximality"

Exact categories: congruences $\leftrightarrow$ quotients



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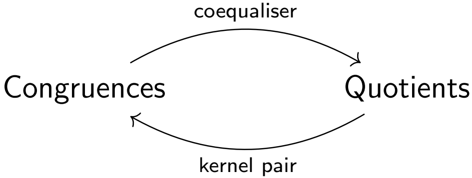

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- KH is an exact category: Manes' Theorem (1967)
- KH is a **pretopos** = coherent + positive + exact

The main result

Theorem (Marra, R.)

Suppose **C** *is a non-trivial category which is a*

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2. *filtral*
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Then **C** *is equivalent to* KH.

Idea of the construction

$\mathrm{pt} = \hom_{\mathbf{C}}(\mathbf{1}, -) \colon \mathbf{C} \to \mathrm{Set}$ faithful functor

$$\begin{array}{c} \mathbf{C} \\ \Big\downarrow {\scriptstyle \mathrm{pt}} \\ \mathrm{Set} \end{array}$$

well-pointed

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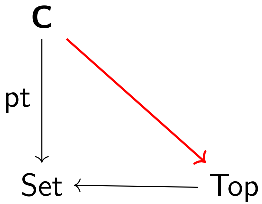

$+$ coherent $+$ positive ($+$ mono-complete)

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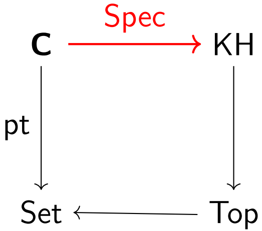

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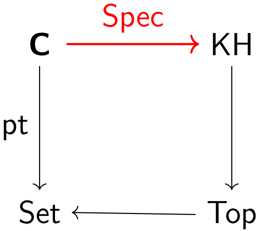

$+$ filtrality

$\mathrm{Spec} \colon \mathbf{C} \to \mathrm{KH}$ is an **equivalence** iff $\mathbf{C}$ is exact and contains $\coprod_J \mathbf{1}$ for every set J.

Final remarks:

- **Boolean spaces** (=zero-dimensional KH spaces) can be characterised in a similar manner, dropping exactness and requiring that every object be filtral
- The "axiomatisation" of KH can be exploited to show that the **exact completion** of Boolean spaces coincides with KH
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- **Order-topological** context? **Point-free** approach? And more...

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For more details: Marra, R., A characterisation of the category of compact Hausdorff spaces, arXiv:1808.09738, submitted.

Thank you for your attention!

Theorem (Richter, 1992)

Suppose $\mathbf{C}$ *is a category admitting an object* $\mathbf{1}$ *such that:*

1. $\mathbf{C}$ *has all set-indexed copowers of* $\mathbf{1}$*;*
2. $\mathbf{1}$ *is a regular generator in* $\mathbf{C}$*;*
3. $\mathbf{C}$ *admits all coequalisers and it is exact;*
4. a $\hom_{\mathbf{C}}(\mathbf{1}, \mathbf{2}) = \{\bot \neq \top\}$, *where* $\mathbf{2} = \mathbf{1} + \mathbf{1}$*;*
   b *for every set* I, *the co-diagonal morphisms*

   $$\sum_I \mathbf{1} \longleftarrow \sum_I \mathbf{1} + \sum_I \mathbf{1} \cong \sum_I \mathbf{2} \longrightarrow \mathbf{2}$$

   are jointly monic;

   c $\mathbf{2}$ *is a coseparator for the full subcategory of* $\mathbf{C}$ *on the set-indexed copowers of* $\mathbf{1}$*;*
5. *there is* $o \in \hom_{\mathbf{C}}(\mathbf{1}, \mathbf{2})$ *such that, for every set* I, $\sum_I \mathbf{1}$ *is* $(\mathbf{1}, o)$*-compact.*

Then $\mathbf{C}$ *is equivalent to* KH*.*