Norm complete Abelian ℓ-groups: topological duality

Joint work with M. Abbadini and V. Marra

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Remark

An answer was already given by Stone: compact Hausdorff spaces correspond to Archimedean, norm-complete and divisible $u\ell$-groups.

Norm-complete ℓ-groups

Definition

A $u\ell$-group $\langle G, +, -, 0, \wedge, \vee, 1\rangle$ is an (Abelian) lattice-ordered group with unit. I.e.,

- $\langle G, +, -, 0\rangle$ is an Abelian group,
- $\langle G, \wedge, \vee\rangle$, is a lattice,
- the operation $+$ is order-invariant: $x \leqslant y \Rightarrow x + z \leqslant y + z$,

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- the constant 1 is an order unit: for all $x \in G$, there exists $n \in \mathbb{N}$ s.t. $x \leqslant (n)1$.

It is customary to write $(n)x := \underbrace{x + \cdots + x}_{n \text{ times}}$ and $|x| := x \vee -x$.

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Let (G, u) be a $u\ell$-group. The order unit 1 induces a seminorm $\| \; \|$ defined as follows:

$$\|g\| := \texttt{inf}\left\{\frac{p}{q} \in \mathbb{Q} \mid p, q \in \mathbb{N}, q \neq 0 \text{ and } (q)|g| \leqslant (p)1\right\}$$

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A $u\ell$-group G is Archimedean:

for all $x, y \in G$ such that $x \geqslant 0$ and $y \geqslant 0$ we have:

if, for all $n \in \mathbb{N}$, $(n)x \leqslant y$, then $x = 0$.

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Definition

A norm complete ℓ-group is an Archimedean, $u\ell$-group that is complete w.r.t. to the norm $\| \,\|$. Morphisms of norm-complete ℓ-groups are functions that preserve $+, \vee, \wedge, -, 0, 1$. This category will by indicated by $\overline{\mathsf{G}}$.

Functional representation

Theorem (Goodearl-Handelman 1980)

Let X be a compact Hausdorff space. For each $x \in X$ choose A_x to be either $A_x = \mathbb{R}$ or $A_x = \frac{1}{n}\mathbb{Z}$.

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The aim of this talk is to make the above functional representation into a categorical duality.

Abstract and real denominators

We can encode the $(A_x)_{x \in X}$ of Goodearl-Handelman via a function $\zeta \colon X \to \mathbb{N}$.

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It is useful to think of $\zeta(x)$ as the (abstract) denominator of x.

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It is useful to think of $\zeta(x)$ as the (abstract) denominator of x. Indeed, saying that

$$f(x) \in A_x,$$

as in the statement of Goodearl-Handelman, amounts to saying that

the (real) denominator of $f(x) \in \mathbb{R}$ divides the (abstract) denominator $\zeta(x)$.

where, if $r \in \mathbb{R} \setminus \mathbb{Q}$, we set $\mathtt{den}(r) = 0$.

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We call a-space a compact Hausdorff space X together with an arbitrary map $\zeta\colon X \to \mathbb{N}$. An a-map from an a-space (X, ζ) into an a-space (Y, ζ') is a continuous map $f\colon X \to Y$ such that $\forall x \in X$,

$$\zeta'(f(x)) \mid \zeta(x) \qquad \textit{f respects the (abstract) denominators}.$$

The category of a-spaces with a-maps is indicated by A.

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Let I be a set and $\overline{p} \in \mathbb{R}^I$. We define the denominator of $\overline{p}$ to be be the following (natural number):

- If $\overline{p} \in \mathbb{Q}^I$ then

$$\texttt{den}(\overline{p}) = \texttt{lcd}\{p_i \mid i \in I\}$$

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Remark

For any set I, for any K closed subset of $\mathbb{R}^I$, the pair $(K, \mathtt{den}_{\restriction_K})$ is an a-space.

The main adjunction

The functor $\mathscr{C}_\zeta$

Let $\mathscr{C}_\zeta \colon \mathsf{A} \to \overline{\mathsf{G}}$ be the assignment that associates to every object $\langle X, \zeta \rangle$ in A the norm-complete ℓ-group

$$\mathscr{C}_\zeta(\langle X, \zeta \rangle) := \{f \colon X \to \mathbb{R} \mid f \text{ cont.}, \forall x \in X \ \mathtt{den}(f(x)) \mid \zeta(x)\},$$

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and to any a-map $g \colon \langle X, \zeta \rangle \to \langle Y, \zeta' \rangle$ the $\overline{\mathsf{G}}$-arrow that sends each $h \in \mathscr{C}_\zeta(\langle Y, \zeta' \rangle)$ into the map $h \circ g$.

The main adjunction

The functor $\mathscr{M}$
Let $\mathscr{M} : \overline{\mathsf{G}} \to \mathsf{A}$ be the assignment that associates to each norm-complete ℓ-group G, the pair $\langle \mathtt{Max}(G), \zeta_G \rangle$, where $\mathtt{Max}(G)$ is maximal spectrum of G and, for any $\mathfrak{m} \in \mathtt{Max}(G)$,

$$\zeta_G(\mathfrak{m}) := \begin{cases} n & \text{if } G/\mathfrak{m} \cong \frac{1}{n}\mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

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Let also $\mathscr{M}$ assign to every $\overline{\mathsf{G}}$-homomorphism $h\colon G \to H$ the map that sends every $\mathfrak{m} \in \mathscr{M}(H)$ into its inverse image under h, in symbols $\mathscr{M}(h)(\mathfrak{m}) = h^{-1}[\mathfrak{m}] \in \mathtt{Max}(G)$.

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Theorem
The functors $\mathscr{C}_\zeta$ and $\mathscr{M}$ form a contravariant adjunction.

Fixed points

To find a (contravariant) categorical equivalence, we are now interested in finding the fixed points of the this adjunction. Namely,

- The $u\ell$-groups G such that $G \cong \mathscr{C}_\zeta \mathscr{M}(G)$ and
- The a-spaces (X, ζ) such that $(X, \zeta) \cong \mathscr{M} \mathscr{C}_\zeta(X, \zeta)$.

Stone-Weierstrass for $u\ell$-groups

The main ingredient to characterise the fixed points of $\mathscr{C}_\zeta \mathscr{M}$ is the following result, which has an interest in its own.

Theorem (Stone-Weierstrass for $u\ell$-groups)

Let (X, ζ) be an a-space, and let $G \subseteq \mathscr{C}_\zeta(X)$ be a $u\ell$-subgroup. Suppose the following hold.

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Corollary

For any norm-complete ℓ-group G one has $G \cong \mathscr{C}_\zeta \mathscr{M}(G)$.

Representable a-spaces

To characterise the fixed points of $\mathscr{M}\mathscr{C}_\zeta$ we preliminary notice that:

Lemma

For any a-space (X, ζ) one has $(X, \zeta) \cong \mathscr{M}\mathscr{C}_\zeta(X, \zeta)$ if, and only if, there exists $K \subseteq \mathbb{R}^I$, closed subspace for some index set I, such that (X, ζ) and $(K, \mathtt{den}_{\restriction_K})$ are A*-isomorphic.*

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So the problem reduces to find a characterisation of the abstract denominators $\zeta \colon X \to \mathbb{N}$ which are concrete denominators for some $(K \subseteq \mathbb{R}^I, \mathtt{den}_{\restriction_K})$.

An easy counter example

Consider $[a, b] \subseteq \mathbb{R}$ with its Euclidean topology and endow it with a constant ζ:

$$\forall x \in [a, b] \quad \zeta(x) = 1.$$

The only points with denominator equal 1 in $\mathbb{R}^I$ are the so-called lattice points i.e., points with integer coordinates.

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The only way an embedding of $([a, b], \zeta)$ could respect ζ is either to send all points in one lattice point —failing injectivity— or by sending the points in different lattice points —failing continuity.

Separating points with functions

It is not hard to see that, in order to a-embed an a-space (X, ζ) into some $(\mathbb{R}^I, \texttt{den})$, we need to have enough $\mathbb{R}$-valued a-maps on X.

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But we want to guarantee this property by enforcing some sort of aritmetico-topological property on the space (X, ζ)!

1.5.11. THEOREM (URYSOHN'S LEMMA). *For every pair A, B of disjoint closed subsets of a normal space X there exists a continuous function $f\colon X \to I$ such that $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \in B$.*

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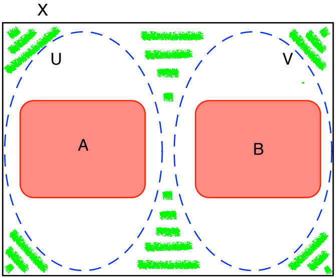
(N2) For any two disjoint closed subsets A and B of X, there exist two disjoint open sets U and V containing A and B, respectively



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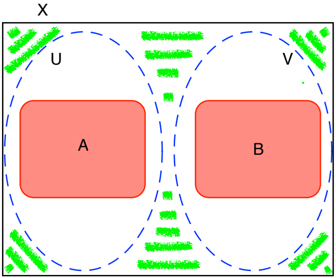
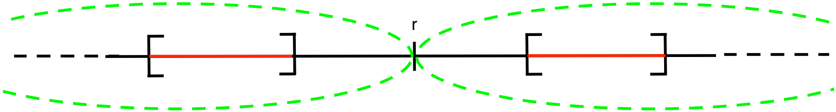


Examples of a-normal spaces



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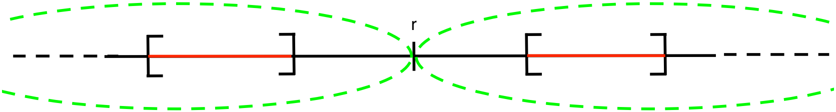

Theorem

For any set I, for any K closed subset of $\mathbb{R}^I$, the a-space $(K, \mathtt{den}_{\restriction_K})$ is a-normal.

Urysohn's lemma for a-spaces

Definition

Let (X, ζ) be an a-space and $x \in X$. A point $\alpha \in \mathbb{R}$ is said to be admissible for x if $\texttt{den}(\alpha)$ divides $\zeta(x)$.

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Let X be an a-normal space. Let A and B be disjoint closed subsets of X. Let $\alpha, \beta \in \mathbb{R}$.

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1. *α is admissible for every point of A,*
2. *β is admissible for every point of B,*

Then, there exists an a-map $f : X \to \mathbb{R}$ such that

$$\text{for all } x \in A, f(x) = \alpha \text{ and for all } y \in B, f(y) = \beta.$$

PROOF STRATEGY

Let X be an a-space, $[\alpha, \beta] \subseteq \mathbb{R}$, and $f\colon X \to [\alpha, \beta]$ be an a-map.
Pick $D \subseteq [\alpha, \beta]$ such that $\alpha, \beta \in D$.

Proof strategy

Let X be an a-space, $[\alpha, \beta] \subseteq \mathbb{R}$, and $f : X \to [\alpha, \beta]$ be an a-map. Pick $D \subseteq [\alpha, \beta]$ such that $\alpha, \beta \in D$. For each $r \in D$, set

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$$A_r \coloneqq f^{-1}[[\alpha, r]], \text{ and } B(r) \coloneqq f^{-1}[[r, \beta]].$$

Then, for all $r, s \in D$, the following properties hold.

1. $B_\alpha = A_\beta = X$.
2. $A_r \cup B_r = X$.
3. $r < s \Rightarrow D, A_r \cap B_s = \emptyset$.
4. $r \leqslant s \Rightarrow$ for every $x \in B_r \cap A_s$ there is $\gamma \in [r, s]$ such that γ is admissible for x.

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Definition

Let X, α, β be as above. An $[\alpha, \beta]$-draft on X consists of a subset $D \subseteq [\alpha, \beta]$ containing α and β and a pair of closed subsets of X (A_r, B_r) for every $r \in D$, such that (1)–(4) hold.

Proof strategy, continued

Let $(D, (A_d, B_d)_{d \in D})$ be a $[\alpha, \beta]$-draft on X.

Proof strategy, continued

Let $(D, (A_d, B_d)_{d \in D})$ be a $[\alpha, \beta]$-draft on X.

- If D is closed, then there exists $(D', (A_d, B_d)_{d \in D'})$ $[\alpha, \beta]$-draft on X that refines $(D, (A_d, B_d)_{d \in D})$, and such that D' is dense in $[\alpha, \beta]$.

$\square$

Proof strategy, continued

Let $(D, (A_d, B_d)_{d\in D})$ be a $[\alpha, \beta]$-draft on X.

- If D is closed, then there exists $(D', (A_d, B_d)_{d\in D'})$ $[\alpha, \beta]$-draft on X that refines $(D, (A_d, B_d)_{d\in D})$, and such that D' is dense in $[\alpha, \beta]$.
- A realisation of an $[\alpha, \beta]$-draft on X $(D, (A_d, B_d)_{d\in D})$ is an a-map $f\colon X \to [\alpha, \beta]$ such that, for every $r \in D$,

$$f[A_r] \subseteq [\alpha, r], \text{ and } f[B(r)] \subseteq [r, \beta].$$

□

Proof strategy, continued

Let $(D, (A_d, B_d)_{d \in D})$ be a $[\alpha, \beta]$-draft on X.

- If D is closed, then there exists $(D', (A_d, B_d)_{d \in D'})$ $[\alpha, \beta]$-draft on X that refines $(D, (A_d, B_d)_{d \in D})$, and such that D' is dense in $[\alpha, \beta]$.
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$$f[A_r] \subseteq [\alpha, r], \text{ and } f[B(r)] \subseteq [r, \beta].$$

- If D is dense in $[\alpha, \beta]$. Then, $(D, (A_d, B_d)_{d \in D})$ has a unique realisation given by

$$f(x) := \texttt{inf}\{r \in D \mid x \in A_r\} = \texttt{sup}\{r \in D \mid x \in B(r)\}.$$

□

Proof strategy, continued

Let $(D, (A_d, B_d)_{d \in D})$ be a $[\alpha, \beta]$-draft on X.

- If D is closed, then there exists $(D', (A_d, B_d)_{d \in D'})$ $[\alpha, \beta]$-draft on X that refines $(D, (A_d, B_d)_{d \in D})$, and such that D' is dense in $[\alpha, \beta]$.
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- Finally, in the stament of the theorem, set $D := \{\alpha, \beta\}$, $A(\alpha) := A$, $B(\alpha) := X$, $A(\beta) := X$, and $B(\beta) := B$.

□

The duality

Theorem

The functors $\mathscr{M}$ and $\mathscr{C}_\zeta$ give a categorical duality between norm-complete ℓ-groups with unit preserving homomorphisms and a-normal spaces with a-maps.

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Thank you for your attention!