Orders on Groups
an approach through spectral spaces

Almudena Colacito
Based on joint work with Vincenzo Marra

Mathematisches Institut
Universität Bern

TACL 2019
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The Spectral Space

The *spectral space*, or ℓ-*spectrum*, was introduced for Abelian ℓ-groups by Klaus Keimel in his doctoral dissertation (1971).

His aim was to follow on from the success of scheme theory in algebraic geometry, and introduce sheaf-theoretic methods in the study of ℓ-groups.

The spectral space has been widely studied for Abelian ℓ-groups, and later introduced also for arbitrary ℓ-groups (e.g. Conrad & Martinez, 1990).

Lattice-Ordered Groups

An ℓ-group is an algebra $\langle \mathrm{H}, \cdot, \wedge, \vee, ^{-1}, e \rangle$ where $\langle \mathrm{H}, \cdot, ^{-1}, e \rangle$ is a group, $\langle \mathrm{H}, \wedge, \vee \rangle$ is a (distributive) lattice, and $\cdot$ distributes over $\wedge, \vee$.

The class of all ℓ-groups is a variety, that is, an *equationally definable class*.

Example

Given a chain Ω, the group $\mathrm{Aut}(\Omega)$ of its *order-preserving bijections* can be made into an ℓ-group by defining the *coordinate-wise lattice order*:

$$f \leq g \iff f(a) \leq g(a), \quad \text{for every } a \in \Omega.$$

The ℓ-groups of order-preserving bijections of chains *generate the variety of ℓ-groups*. More precisely, the ℓ-group $\mathrm{Aut}(\mathbb{R})$ *generates the variety*.

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The Spectral Space

A convex ℓ-subgroup of an ℓ-group H is a *convex sublattice subgroup* of H.

For a convex ℓ-subgroup $\mathfrak{c} \subseteq \mathrm{H}$, the quotient $\mathrm{H}/\mathfrak{c}$ is a *lattice* with operations $\mathfrak{c}x \wedge \mathfrak{c}y = \mathfrak{c}(x \wedge y)$, and $\mathfrak{c}x \vee \mathfrak{c}y = \mathfrak{c}(x \vee y)$.

We call a convex ℓ-subgroup $\mathfrak{p} \subsetneq \mathrm{H}$ prime if for every $x, y \in \mathrm{H}$, if $x \wedge y \in \mathfrak{p}$ then $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. Equivalently, if the quotient $\mathrm{H}/\mathfrak{p}$ *is a chain*.

The set $\operatorname{Spec} \mathrm{H}$ of *prime convex ℓ-subgroups* ordered by $\subseteq$ is a *root system*.

Every $\mathfrak{p} \in \operatorname{Spec} \mathrm{H}$ contains at least one *minimal* element $\mathfrak{m} \in \operatorname{Min} \mathrm{H}$ (by a simple application of Zorn's Lemma).

The Spectral Space

We endow $\operatorname{Spec} \mathrm{H}$ with the spectral topology, whose basic open sets are

$$\mathbb{S}(x) = \{\mathfrak{p} \in \operatorname{Spec} \mathrm{H} \mid x \notin \mathfrak{p}\}, \quad \text{for } x \in \mathrm{H}.$$

These are all the *compact opens*, and the space is a completely normal generalised spectral space.

We endow $\operatorname{Min} \mathrm{H}$ with the subspace topology, and get a Hausdorff space. It is not necessarily *compact*.

Example: Free Abelian ℓ-Group

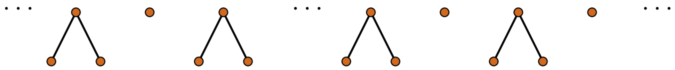
Consider the ℓ-group of piece-wise homogeneous linear functions $f \colon \mathbb{R}^2 \to \mathbb{R}$ with integral coefficients.

It is generated *as a distributive lattice* by $\mathbb{Z}^2$.

It is the free Abelian ℓ-group $\mathsf{F}^{\ell}_{\mathsf{A}}(2)$ over 2 generators.

Example: Free Abelian ℓ-Group

The root system $\mathrm{Spec}\, \mathrm{F}^{\ell}_{\mathsf{A}}(2)$ is:



Every element $\mathfrak{p} \in \mathrm{Spec}\, \mathrm{F}^{\ell}_{\mathsf{A}}(2)$ is extended by a *unique maximal* element $\mathfrak{p}^* \in \mathrm{Max}\, \mathrm{F}^{\ell}_{\mathsf{A}}(2)$, and we can consider the *continuous closed* map

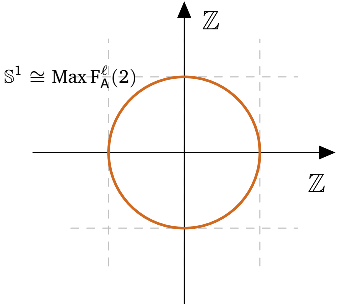
$$\lambda : \mathrm{Min}\, \mathrm{F}^{\ell}_{\mathsf{A}}(2) \to \mathrm{Max}\, \mathrm{F}^{\ell}_{\mathsf{A}}(2),$$

defined by

$$\lambda(\mathfrak{m}) = \mathfrak{m}^*.$$

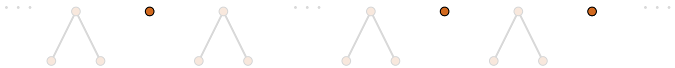
Example: Free Abelian ℓ-Group

The space $\operatorname{Max} \mathrm{F}^{\ell}_{\mathsf{A}}(2)$ is homeomorphic to $\mathbb{S}^1$ with the Euclidean topology.

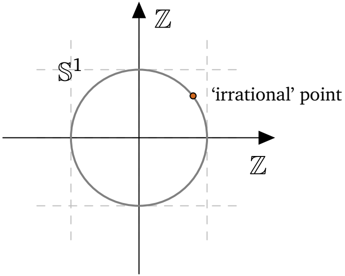


EXAMPLE: FREE ABELIAN ℓ-GROUP

The points in $\mathrm{Max}\,\mathrm{F}^{\ell}_{\mathsf{A}}(2) \cap \mathrm{Min}\,\mathrm{F}^{\ell}_{\mathsf{A}}(2)$

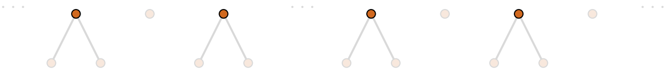


can be thought of as those points $(x, y) \in \mathbb{S}^1$ with $\frac{x}{y} \notin \mathbb{Q}$.



Example: Free Abelian ℓ-Group

The points in $\operatorname{Max} \mathrm{F}^{\ell}_{\mathsf{A}}(2) \setminus \operatorname{Min} \mathrm{F}^{\ell}_{\mathsf{A}}(2)$



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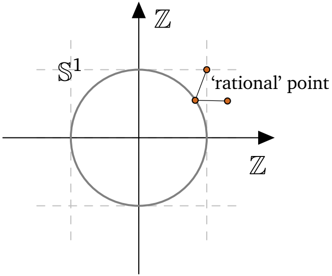

Example: Compactness

Example

Consider the Abelian ℓ-group H_1 generated in $\mathrm{C}([0,1],\mathbb{R})$ by the maps

$$\hat{1}\colon [0,1] \to \mathbb{R},\quad x \mapsto 1 \quad \text{and} \quad \mathrm{id}_{[0,1]}\colon [0,1] \to \mathbb{R},\quad x \mapsto x.$$

The space $\operatorname{Min}\mathrm{H}_1$ is compact.

Non-Example

Consider the Abelian ℓ-group H_2 generated in $\mathrm{C}([0,1],\mathbb{R})$ by the maps

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The space $\operatorname{Min}\mathrm{H}_2$ is *not* compact.

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The space $\operatorname{Min}\mathrm{H}_2$ is *not* compact.

Right Orders

The theory of orderable groups is often presented as an area of the theory of lattice-ordered groups (briefly, ℓ-groups).

Example

A group is right orderable if, and only if, it embeds into an ℓ-group.

Orderability of many interesting groups has recently attracted the interest of people from different areas in mathematics.

Example

Orderability of the fundamental group of a 3-manifold is related to the existence of certain foliations. (Boyer, Rolfsen, & Wiest, 2005)

Example

A countable group is right orderable if, and only if, it acts faithfully on $\mathbb{R}$ by orientation-preserving homeomorphisms.

Right Orders

Given a group $\langle G, \cdot, ^{-1}, e\rangle$ a *right-invariant* (total) order $\leq$ on G is a total order on G such that for every $a, b, t \in G$,

$$a \leq b \Longrightarrow a \cdot t \leq b \cdot t.$$

We call a *right-invariant* order on a group G a (total) right order on G.

Given a right order on G, the set of its *non-negative elements* $C \subseteq G$ is a submonoid of G with the properties $C \cup C^{-1} = G$ and $C \cap C^{-1} = \{e\}$, and we call such a submonoid a (total) cone for G.

Conversely, every *cone* C is the positive cone of some right order $\leq_C$ on G, defined via: $a \leq_C b$ if, and only if, $ba^{-1} \in C$.

We identify a right order $\leq$ on G with its cone C, and hence see the set $\mathcal{R}(G)$ of *all possible right orders on* G as a set of subsets of G.

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The Topological Space of Orders

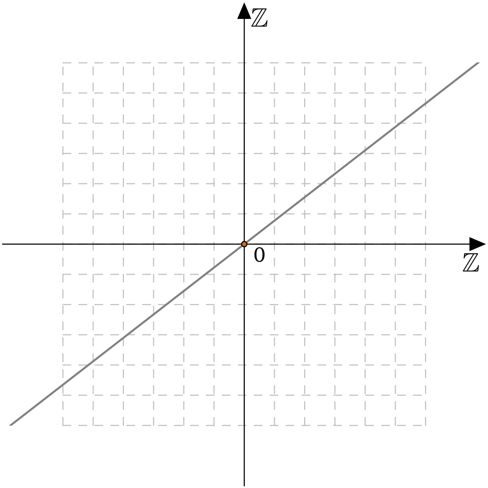
Given a *right orderable* group G, the set $\mathcal{R}(\mathrm{G})$ of right orders on G is non-empty, and it is possible to endow it with a topology τ, namely the *smallest topology* containing the sets

$$\mathbb{R}(a) = \{C \in \mathcal{R}(\mathrm{G}) \mid a \in C\}, \quad \text{for } a \in \mathrm{G}.$$

This space was introduced by Adam Sikora in 2004, and was proved to be compact, Hausdorff, and totally disconnected.

Example: Orders on $\mathbb{Z}^2$

Lines through the origin, with irrational slope...



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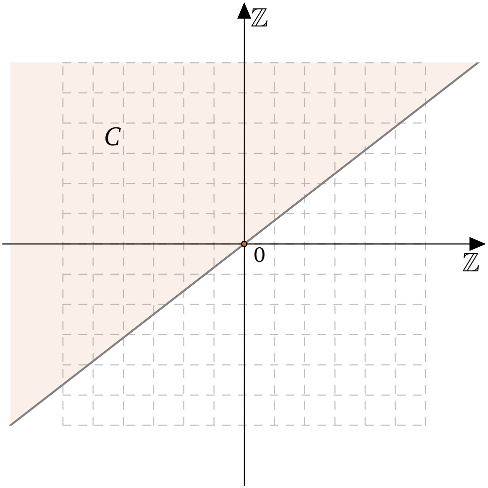
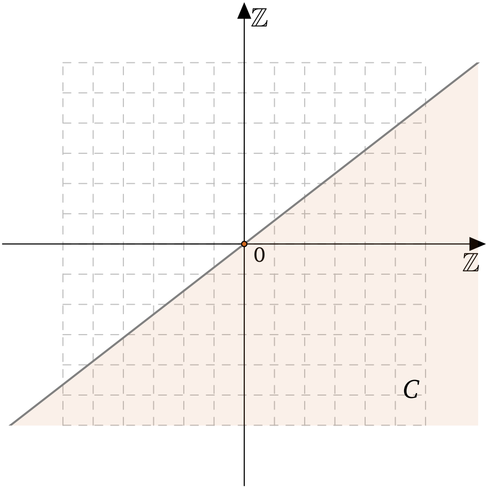
... determine *one* (Archimedean) *order*...



Example: Orders on $\mathbb{Z}^2$

... determine *two* (Archimedean) *orders*.



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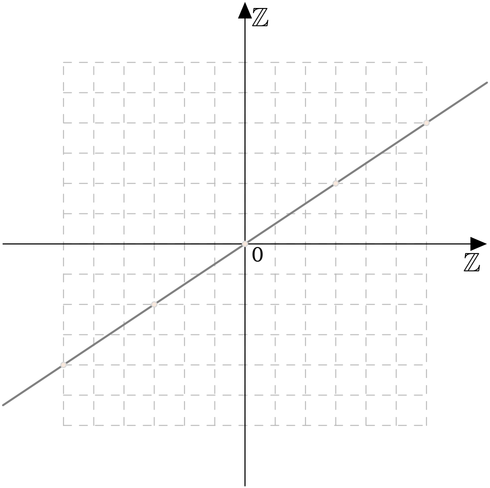

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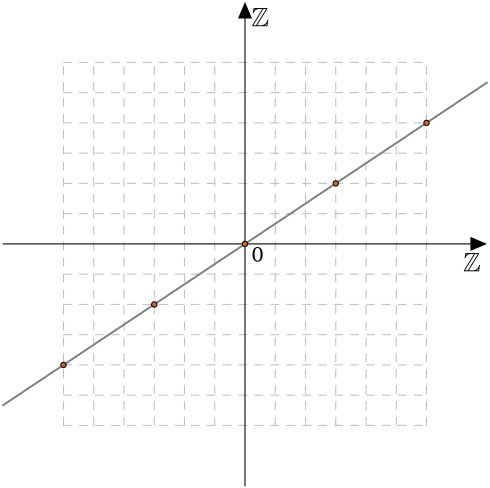

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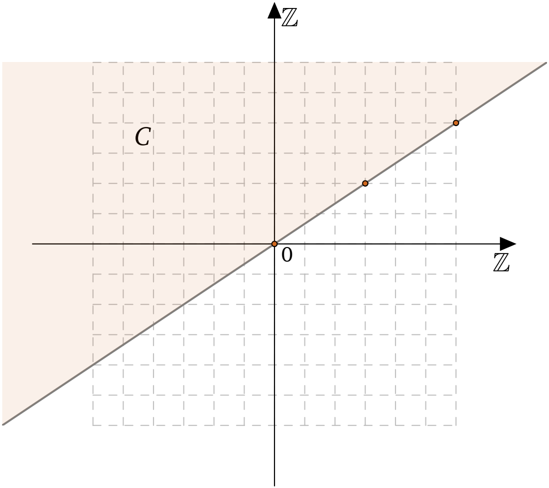
... determine *one* (lexicographic) *order*...



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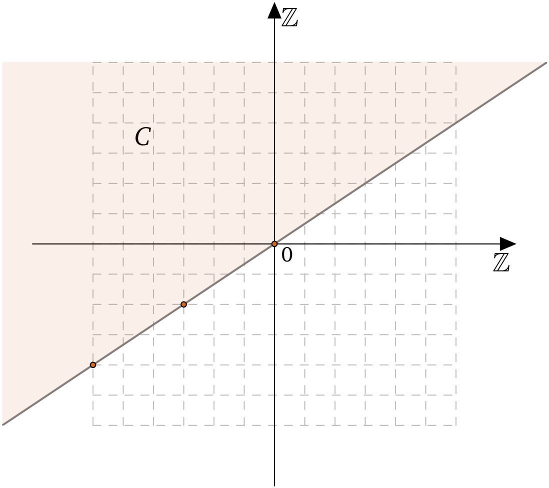
... determine *two* (lexicographic) *orders*...



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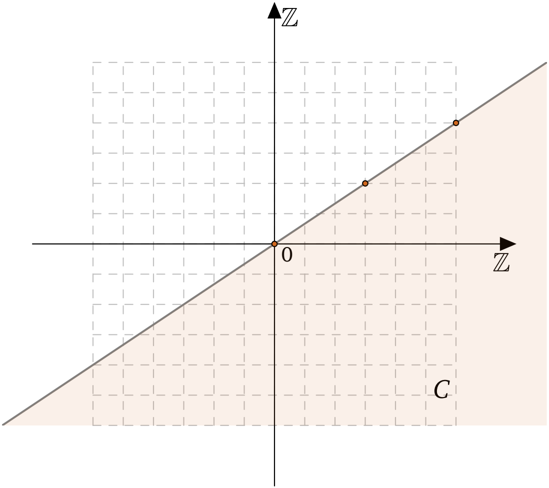
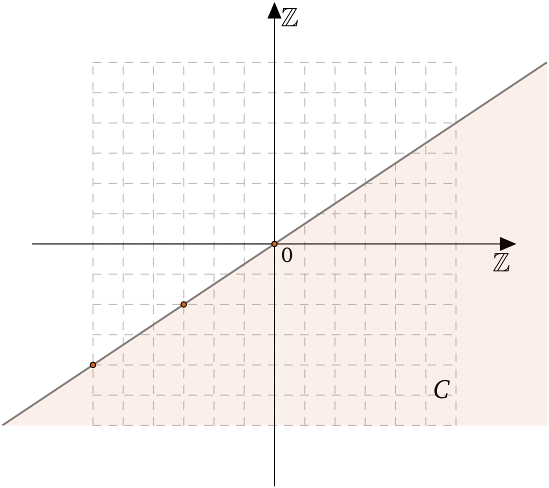


Example: Orders on $\mathbb{Z}^2$

... determine *four* (lexicographic) *orders*.



Example: Isolated Points

Let $\mathrm{F}(n)$ be the free group generated by $n \geq 2$ variables.

Theorem (McCleary)

The space $\mathcal{R}(\mathrm{F}(n))$ for $n \geq 2$, doesn't have any isolated points.

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Given a group $\langle \mathrm{G}, \cdot, {}^{-1}, e \rangle$ an order $\leq$ on G is a *right order* on G which is also *left invariant*, that is, for every $a, b, t \in \mathrm{G}$, $a \leq b \implies t \cdot a \leq t \cdot b$.

We identify an order $\leq$ on G with its cone C, and hence see the space $\mathcal{O}(\mathrm{G})$ of *all possible orders on G* as a subspace of $\mathcal{R}(\mathrm{G})$.

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Conjecture.

The space $\mathcal{O}(\mathrm{F}(n))$ for $n \geq 2$, doesn't have any isolated points.

This was first asked by McCleary (1986) in a different form.

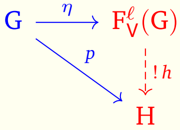
(OPEN) QUESTION. Does G $(1 < \eta < \infty)$ have a finite subset S for which there is a unique (two-sided) total order of G_η making all elements of S positive?

FREE ℓ-GROUPS

For a group G and a variety V of ℓ-groups, there are an ℓ-group $F^{\ell}_{\mathsf{V}}(G)$ and a group homomorphism $\eta\colon G \to F^{\ell}_{\mathsf{V}}(G)$ characterised by the following...

... UNIVERSAL PROPERTY.

For each group homomorphism $p\colon G \to H$ with $H \in \mathsf{V}$, there is exactly one ℓ-homomorphism $h\colon F^{\ell}_{\mathsf{V}}(G) \to H$ such that the following diagram



commutes, i.e., $h(\eta(a)) = p(a)$, for each $a \in G$.

It is easy to see that $\eta[G]$ *generates* $F^{\ell}_{\mathsf{V}}(G)$ as a *lattice*.

Example: Compactness

Theorem

For any right-orderable group G*, the minimal spectrum* $\mathrm{Min}\,\mathrm{F}^{\ell}(\mathrm{G})$ *of the free* ℓ*-group* $\mathrm{F}^{\ell}(\mathrm{G})$ *is the space* $\mathcal{R}(\mathrm{G})$ *of right orders on* G*.*

Since $\mathcal{R}(\eta[\mathrm{G}])$ is very easily proved compact, we get:

Corollary

For any group G*, the space* $\mathrm{Min}\,\mathrm{F}^{\ell}(\mathrm{G})$ *of the free* ℓ*-group* $\mathrm{F}^{\ell}(\mathrm{G})$ *is compact.*

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Back to the Example: Compactness

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The space $\operatorname{Min} \mathrm{H}_2$ is *not* compact.

Example: Isolated Points

The class of those ℓ-groups which are *subdirect products* of totally ordered groups forms the variety R of representable ℓ-groups.

Theorem

The minimal spectrum $\operatorname{Min} \mathrm{F}^{\ell}_{\mathsf{R}}(n)$ *of the free representable* ℓ*-group* $\mathrm{F}^{\ell}_{\mathsf{R}}(n)$ *of rank* n *is the space* $\mathcal{O}(\mathrm{F}(n))$ *of orders on the free group* $\mathrm{F}(n)$ *of rank* n*.*

EXAMPLE: ISOLATED POINTS

Every element $\mathfrak{p} \in \operatorname{Spec} \mathrm{F}^{\ell}_{\mathsf{R}}(n)$ for $n \geq 2$ is extended by a *unique maximal* element $\mathfrak{p}^* \in \operatorname{Max} \mathrm{F}^{\ell}_{\mathsf{R}}(n)$, and we can consider the *continuous closed* map

$$\lambda : \operatorname{Min} \mathrm{F}^{\ell}_{\mathsf{R}}(n) \to \operatorname{Max} \mathrm{F}^{\ell}_{\mathsf{R}}(n),$$

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$$\lambda(\mathfrak{m}) = \mathfrak{m}^*.$$

It is possible to show: $\operatorname{Max} \mathrm{F}^{\ell}_{\mathsf{R}}(n) \cong \operatorname{Max} \mathrm{F}^{\ell}_{\mathsf{A}}(n) \cong \mathbb{S}^{n-1}$.

We say that the map λ is irreducible if it sends *proper closed* subsets of $\operatorname{Min} \mathrm{F}^{\ell}_{\mathsf{R}}(n)$ to *proper closed* subsets of $\operatorname{Max} \mathrm{F}^{\ell}_{\mathsf{R}}(n)$.

COROLLARY

If the map λ is irreducible, then $\operatorname{Min} \mathrm{F}^{\ell}_{\mathsf{R}}(n)$ doesn't have any isolated points.

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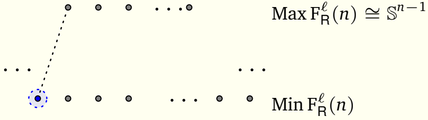
Corollary

If the map λ is irreducible, then $\operatorname{Min} F^{\ell}_{\mathsf{R}}(n)$ doesn't have any isolated points.

Example: Isolated Points

Proof...

Suppose that there is an isolated point in $\mathrm{Min}\,\mathrm{F}^{\ell}_{\mathsf{R}}(n)$.



The image of the green points through the map λ must be *proper*.

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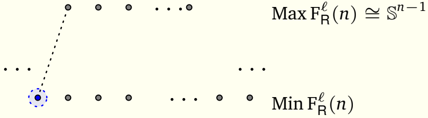

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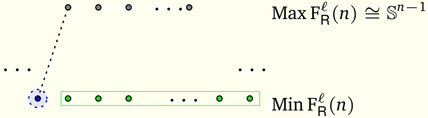
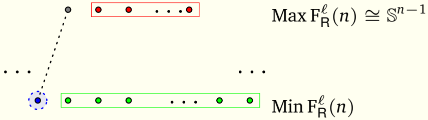


Example: Isolated Points

… Proof.

Hence, the image of the green points through the map λ is the red part.



This is not possible, since $\operatorname{Max} F^{\ell}_{R}(n)$ is homeomorphic to $\mathbb{S}^{n-1}$ with the Euclidean topology and hence, it doesn't have isolated points. ↯

Therefore, if λ is irreducible, then $\operatorname{Min} F^{\ell}_{R}(n)$ doesn't have isolated points. □

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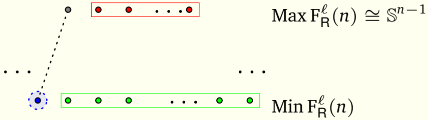
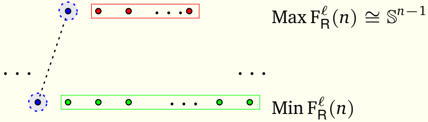


This is not possible, since $\mathrm{Max}\,\mathrm{F}^{\ell}_{\mathsf{R}}(n)$ is homeomorphic to $\mathbb{S}^{n-1}$ with the Euclidean topology and hence, it doesn't have isolated points. $\lightning$



Therefore, if λ is irreducible, then $\mathrm{Min}\,\mathrm{F}^{\ell}_{\mathsf{R}}(n)$ doesn't have isolated points. □

Concluding Remarks

What's next?

Recall that $\operatorname{Max}\mathrm{F}^{\ell}_{\mathsf{R}}(n) \cong \operatorname{Max}\mathrm{F}^{\ell}_{\mathsf{A}}(n) \cong \mathbb{S}^{n-1}$.
The ℓ-group $\mathrm{F}^{\ell}_{\mathsf{R}}(n)$ acts in various ways on $\operatorname{Max}\mathrm{F}^{\ell}_{\mathsf{R}}(n) \cong \mathbb{S}^{n-1}$.
We seek a *representation* of $\mathrm{F}^{\ell}_{\mathsf{R}}(n)$ in $\mathrm{Homeo}(\mathbb{S}^{n-1})$.
Possibly, exploiting the *dynamic realisation* of orderable groups.

A. Colacito and V. Marra. Orders on groups, and spectral spaces of lattice-groups. arXiv Preprint available. Submitted (2019).

Thank You For Your Attention

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