Difference hierarchies over lattices[1]

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(based on joint work with Gerhke, Krebs, and Straubing)

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Topology, **A**lgebra and **C**ategories in **L**ogic

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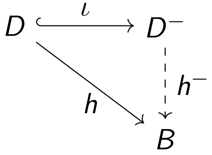
1. Introduction
2. Difference chains of closed upsets
3. The canonical extension approach
4. An application to Logic on Words

D = bounded distributive lattice

Booleanization of D: unique (up to isomorphism) Boolean algebra D^-, together with a bounded lattice embedding $D \overset{\iota}{\hookrightarrow} D^-$ satisfying the following universal property:



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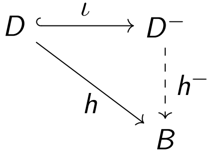

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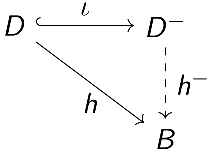

D^- is the unique (up to isomorphism) Boolean algebra containing D as a bounded sublattice and generated as a Boolean algebra by D.

Fact: Every element of D^- may be written as a difference chain of the form

$$a_1 - (a_2 - (\cdots - (a_{n-1} - a_n))\dots),$$

for some $a_1 \geq \cdots \geq a_n$ in D.

Èsakia: Identifies Heyting algebras with the skeleton of closure algebras, and proves that every element a in a closure algebra may be written as a disjunction

$$a = \bigvee_{k=1}^{m} (a_k - a_{k+1}) \vee a_{m+1},$$

with $a_1 \geq a_2 \geq \cdots \geq a_{m+1}$ constructed from a and using the closure operator.

Priestley spaces[1] $\leftrightsquigarrow$ **Bounded distributive lattices**

X = Priestley space $\rightsquigarrow$ $\mathrm{UpClopen}(X)$

$(X_D, \tau, \leq)$, where $\leftsquigarrow$ D = bounded distributive lattice

- $X_D = \{\text{prime filters of } D\}$
- τ has basis of (cl)opens $\{\widehat{a},\ (\widehat{a})^c \mid a \in D\}$, with $\widehat{a} = \{x \in X_D \mid a \in x\}$
- $\leq$ is inclusion of prime filters

$$D \cong \mathrm{UpClopen}(X_D) \quad \text{and} \quad X \cong X_{\mathrm{UpClopen}(X)}$$

In particular, $D^- \cong \mathrm{Clopen}(X_D)$.

[1] Compact and totally order disconnected topological space

X = Priestley space, $V \subseteq X$ = clopen subset.

Then, there are clopen upsets $W_1 \supseteq \cdots \supseteq W_n$ of X such that

$$V = W_1 - (W_2 - (\cdots - (W_{n-1} - W_n)) \ldots).$$

Our question: Is there a *"canonical form"* for such a writing?

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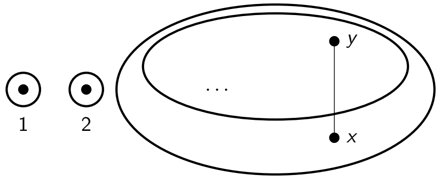
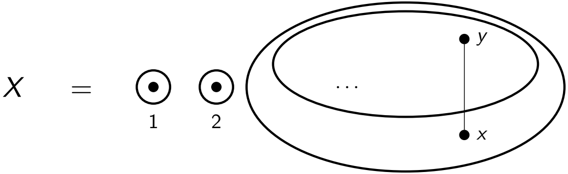
$X =$ 

$\mathsf{UpClopen}(X) = \mathcal{P}_{fin}(\mathbb{N}) \cup \{W \mid W \subseteq X \text{ is cofinite and } y \in W\}$



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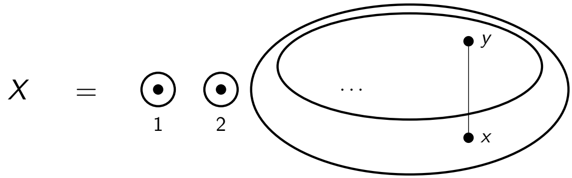
$V = \{x\}$ is clopen



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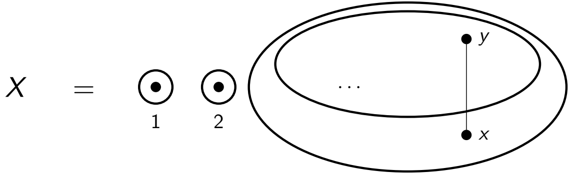
$V = \{x\}$ is clopen, $\quad V = W - W' \implies \uparrow V = \{x, y\} \subseteq W$ is not open!



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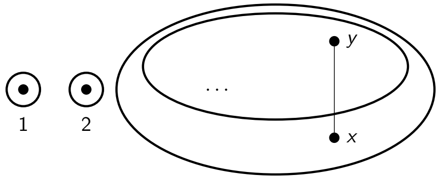
There is no smallest clopen upset containing V:

the clopen upsets containing V are precisely the sets of the form $W = S \cup \{x, y\}$, with $S \subseteq \mathbb{N}$ cofinite.

Moreover, $W' = W - \{x\} = \uparrow(W - V)$ is also a clopen upset and $V = W - W'$.

$X =$ 

$$\mathrm{UpClopen}(X) = \mathcal{P}_{fin}(\mathbb{N}) \cup \{W \mid W \subseteq X \text{ is cofinite and } y \in W\}$$

$V = \{x\}$ is clopen, $\quad V = W - W' \implies \uparrow V = \{x, y\} \subsetneq W$ is not open!

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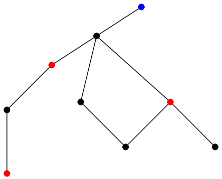
Moreover, $W' = W - \{x\} = \uparrow(W - V)$ is also a clopen upset and $V = W - W'$.

However, $\uparrow V$ is closed and $V = \uparrow V - \uparrow(\uparrow V - V)$.

$P =$ poset, $S \subseteq P$, $p \in P$



$p_1 < p_2 < \cdots < p_n$ in P is an alternating sequence of length n for p (with respect to S) provided

$p_n = p$ and $p_i \in S$ if and only if i is odd.

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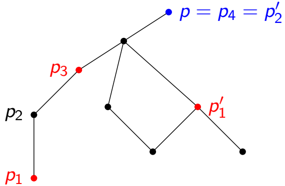

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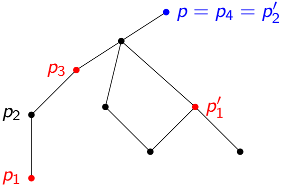

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The degree of p (wrt S), $\deg_S(p)$, is the largest k for which there is an alternating sequence of length k for p,

and p has degree 0 if there is no alternating sequence for p (wrt S).

Example: p has degree 4.

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Remarks:

- The elements of degree 0 are precisely those of $(P - \uparrow S)$.
- An element of finite degree is of odd degree if and only if it belongs to S.
- If S is convex[1], then every element of S has degree 1, while every element of $\uparrow S - S$ has degree 2.

[1] S is convex if $x \leq y \leq z$ with $x, z \in S$ implies $y \in S$.

In general, there are posets where
every element has an infinite degree:

Proposition

X = Priestley space, $V \subseteq X$ = clopen subset.

Then, every element of X has finite degree with respect to V.

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Idea of Proof:

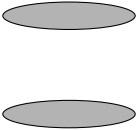
- Any clopen subset of X may be written as a finite union $V = \bigcup_{i=1}^{n}(U_i - W_i)$, with $U_i, W_i \in \text{UpClopen}(X)$.
- (Pigeonhole Principle + convexity of $(U_i - W_i)$) $\implies \deg_V(x) \leq 2n$, for $x \in X$. □

Difference chains of closed upsets

Suppose X = Priestley space, V = clopen subset of X

$$V = G_1 - (G_2 - (\cdots - (G_{n-1} - G_n))\ldots)$$

for some closed upsets $G_1 \supseteq \cdots \supseteq G_n$.



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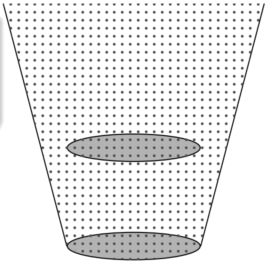
$$V \subseteq G_1 \implies \uparrow V \subseteq G_1$$



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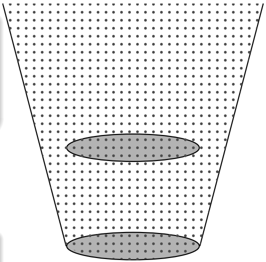

$$V \subseteq G_1 \implies \uparrow V \subseteq G_1$$

$K_1 = \uparrow V$ is the smallest possible choice for G_1, and

$$K_1 = \{x \in X \mid \deg_V(x) \geq 1\}.$$

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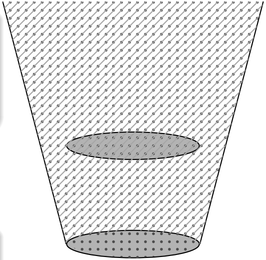

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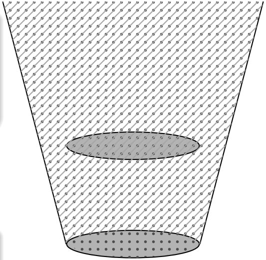

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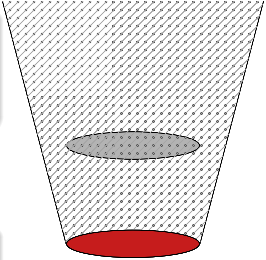

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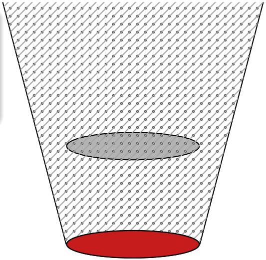

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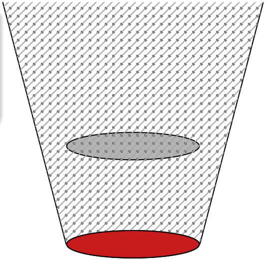

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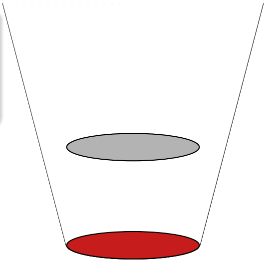

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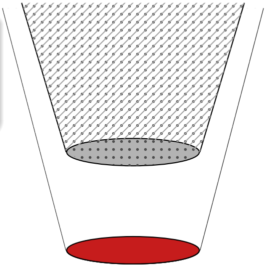

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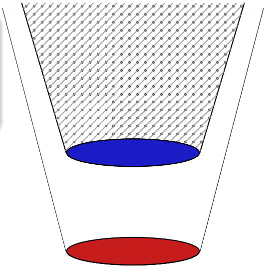

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Also, $\deg_{V'}(x) = \deg_V(x) - 2$, thus $K_i = \{x \in X \mid \deg_V(x) \geq i\}$ ($i = 3, 4$),

and $K_3 - K_4 = \{x \in X \mid \deg_V(x) = 3\}$.

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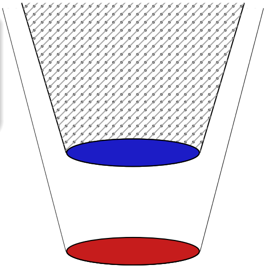

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Theorem

X = Priestley space, V = clopen subset of X, define:

$$K_1 = \uparrow V, \quad K_{2i} = \uparrow(K_{2i-1} - V), \quad K_{2i+1} = \uparrow(K_{2i} \cap V).$$

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$$V = \bigcup_{i=1}^{m} (K_{2i-1} - K_{2i}) = K_1 - (K_2 - (\cdots - (K_{2m-1} - K_{2m}))\dots),$$

where $2m - 1 = \max\{\deg_V(x) \mid x \in V\}$.

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$$V = \bigcup_{i=1}^{m} (K_{2i-1} - K_{2i}) = K_1 - (K_2 - (\cdots - (K_{2m-1} - K_{2m}))\dots),$$

where $2m - 1 = \max\{\deg_V(x) \mid x \in V\}$.

Moreover, if $G_1 \supseteq G_2 \supseteq \cdots \supseteq G_{2p}$ is a chain of closed upsets satisfying

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$$p \geq m, \qquad K_i \subseteq G_i, \; (i \geq 1) \qquad \bigcup_{i=1}^{n} (G_{2i-1} - G_{2i}) \subseteq \bigcup_{i=1}^{n} (K_{2i-1} - K_{2i}) \; (n \geq 1)$$

Recall: A co-Heyting algebra is a bounded distributive lattice D equipped with a binary operation $\_/\_$ such that for every $a \in D$,
$(\_/a)$ is lower adjoint of $(a \vee \_)$: $(x/a \leq b \iff x \leq a \vee b)$.

Fact

A bounded distributive lattice D admits a co-Heyting structure if and only if its Booleanization is equipped with a ceiling function

$$D^- \longrightarrow D, \quad b \mapsto \lceil b \rceil = \bigwedge\{c \in D \mid b \leq c\}.$$

When that is the case, taking upsets preserves clopens of the dual X_D and the functions

$$\lceil \_ \rceil : D^- \to D \quad \text{and} \quad \uparrow_{\_} : \mathrm{Clopen}(X_D) \to \mathrm{UpClopen}(X_D)$$

are naturally isomorphic.

Corollary

$D =$ co-Heyting algebra, $\quad b \in D^-$.

Define:

$$a_1 = \lceil b \rceil, \qquad a_{2i} = \lceil a_{2i-1} - b \rceil, \qquad \text{and} \qquad a_{2i+1} = \lceil a_{2i} \wedge b \rceil,$$

for $i \geq 1$.

Then, the sequence $\{a_i\}_{i \geq 0}$ is decreasing, and there exists $m \geq 1$ such that $a_{2m+1} = 0$ and

$$b = a_1 - (a_2 - (\dots(a_{2m-1} - a_{2m})\dots)),$$

and this is a canonical writing!

- Every finite distributive lattice is a co-Heyting algebra.

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- Every bounded distributive lattice is the direct limit of its finite sublattices.
- Booleanization commutes with direct limits of bounded distributive lattices: $(\varinjlim D_i)^- = \varinjlim D_i^-$.

Corollary

Every Boolean element over any bounded distributive lattice may be written as a difference chain of elements of the lattice.

The canonical extension approach

Recall: If D is a bounded distributive lattice, its canonical extension is an embedding $D \hookrightarrow D^\delta$ into a complete lattice D^δ such that:

- D is dense in D^δ, ie, each element of D^δ is a join of meets and a meet of joins of elements of D;
- the embedding is compact, ie, for every $S, T \subseteq D$, if $\bigwedge S \leq \bigvee T$, then there are finite subsets $S' \subseteq S$ and $T' \subseteq S$ so that $\bigwedge S' \leq \bigvee T'$.

The filter elements of D^δ, $F(D^\delta)$, are those in the meet-closure of D.

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Set $B = D^-$, $X =$ Priestley space of D.

- $F(D^\delta) \cong \mathit{UpClosed}(X)$ and $F(B^\delta) \cong \text{Closed}(X)$.
- $D \hookrightarrow B$ extends to a complete embedding $D^\delta \hookrightarrow B^\delta$.
- This embedding has a lower adjoint $\overline{(\_)} : B^\delta \to D^\delta$ given by $\overline{u} = \min\{v \in D^\delta \mid u \leq v\}$, which preserves filter elements.

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- This embedding has a lower adjoint $\overline{(\_)} : B^\delta \to D^\delta$ given by $\overline{u} = \min\{v \in D^\delta \mid u \leq v\}$, which preserves filter elements.

In particular, $\overline{(\_)} : F(B^\delta) \to F(D^\delta)$ and $\uparrow\_ : Closed(X) \to UpClosed(X)$ are naturally isomorphic.

Our previous result may be stated as follows:

Theorem

D = bounded distributive lattice, $b \in D^-$, define

$$k_1 = \overline{b}, \qquad k_{2n} = \overline{k_{2n-1} - b}, \qquad k_{2n+1} = \overline{k_{2n} \wedge b}.$$

Then,

$$b = k_1 - (k_2 - (\ldots(k_{2n-1} - k_{2n}))\ldots).$$

B = Boolean algebra, I = chain,

$\{S_i\}_{i \in I}$ = increasing chain of meet-subsemilattices of B,

st: $D = \bigcup_{i \in I} S_i$ is a bounded sublattice of B, and each inclusion $g_i : S_i \hookrightarrow B$ admits an upper adjoint $f_i : B \to S_i$.

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Proposition

- $\overline{(\_)}^i = g_i f_i : B \to B$ is a closure operator,
- for every $x \in B$, we have $\overline{x} = \bigwedge_{i \in I} \overline{x}^i$, where the meet is taken in B^δ.

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Theorem

For $b \in B$, define

$$c_{1,i} = \overline{b}^i, \qquad c_{2k,i} = \overline{c_{2k-1,i} - b}^i, \qquad c_{2k+1,i} = \overline{c_{2k,i} \wedge b}^i$$

If $b \in D^- \subseteq B$, then there is $n \in \mathbb{N}$, $i \in I$ so that, for every $j \geq i$ we have

$$b = c_{1,j} - (c_{2,j} - (\cdots - (c_{2n-1,j} - c_{2n})) \ldots).$$

Corollary

$B =$ Boolean algebra, $\qquad I =$ chain,

$\{S_i\}_{i \in I} =$ increasing chain of meet-subsemilattices of B,

st: $D = \bigcup_{i \in I} S_i$ is a bounded sublattice of B, and each inclusion $g_i : S_i \hookrightarrow B$ admits an upper adjoint $f_i : B \to S_i$.

$B' \leq B =$ Boolean subalgebra closed under $\overline{(\_)}^i = g_i f_i$ for $i \in I$. Then,

$$(D \cap B')^- = D^- \cap B', \tag{1}$$

(the Booleanization of a sublattice of B is the Boolean subalgebra of B it generates)

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Open question

What are necessary conditions so that (1) holds?

An application to Logic on Words

Skip a bit Skip all

k-ary numerical predicate $=$ subset R of $\mathbb{N}^k$

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For each set $\mathcal{N}$ of numerical predicates, a word $u \in A^+$ may be thought of as a relational structure

$$\mathcal{M}_u = (\{1, 2, \ldots, |u|\}, (R)_{R \in \mathcal{N}}, (\mathbf{a})_{a \in A})$$

where $\mathbf{a}$ is interpreted as the set of integers i such that the i-th letter of u is an a, and R as $R \cap \{1, \ldots, |u|\}$.

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Example

Let $\mathcal{N} = \{<\}$, where $< = \{(i, j) \mid i < j\}$ is the usual order relation.
For $u = abbaab$, we have

$$\mathcal{M}_u = (\{1, 2, 3, 4, 5, 6\}, <, (\mathbf{a}, \mathbf{b}))$$

with $\mathbf{a} = \{1, 4, 5\}$, $\mathbf{b} = \{2, 3, 6\}$, and $< = \{(1, 2), (1, 3), \ldots, (5, 6)\}$.

The formula $\phi = \exists x \; \mathbf{a}x$ interprets as:

> *There exists a position x in u such that*
> *the letter in position x is an a.*

This defines the language $A^* a A^*$.

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The formula $\forall x \ (x \equiv r \mod n) \to \mathbf{a}x$ defines the language $(A^{r-1}aA^{n-r})^*$.

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The formula $\forall x\ (\mathbf{Prime}(x)\ \vee\ \neg\ \mathbf{a}x)$ defines the language
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$(<)$, $(+1)$, and $(\equiv r \mod n)$ are examples of regular numerical predicates;
$\mathbf{Prime}(x)$ is not regular.

A language L is regular iff the congruence $\sim_L$ given by

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has finite index, or equivalently, iff the Boolean algebra generated by the languages $x^{-1}Ly^{-1} = \{u \in A^+ \mid xuy \in L\}$ $(x, y \in A^*)$ is finite.

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Theorem (Straubing'1991)

A numerical predicate is regular if and only if it is equivalent to a first-order formula in the atomic formulas

$$x < y \qquad \textit{and} \qquad x \equiv 0 \mod n.$$

$\mathcal{N}$ = set of numerical predicates

$\Pi_n[\mathcal{N}]$ = formulas $\forall^+ \exists^+ \cdots \varphi$, with $n-1$ quantifier alternations, and φ a quantifier-free formula using numerical predicates from $\mathcal{N}$

(E.g. $\forall x_1 \forall x_2 \exists x_3 \forall x_4 \varphi$ belongs to $\Pi_3[\mathcal{N}]$)

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Question

$$\mathcal{B}\Pi_n[arb] \cap Reg = \mathcal{B}\Pi_n[Reg] \ ?$$

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$$\mathcal{B}\Pi_n[arb] \cap Reg = \mathcal{B}\Pi_n[Reg] \ ?$$

For $n = 1$, the answer is YES (difficult proof, based on a combination of Semigroup and Ramsey Theory).

For $n > 1$, this is still an open problem.

Using the corollary of our results on difference chains, we can give a simple proof of the $n = 1$ case:

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Idea:

1. Take $B = \mathcal{P}(A^+)$, $B' = Reg$, and $S_n = \Pi_1^n[arb]$ (i.e., formulas $\forall x_1 \ldots \forall x_n\ \varphi$).
2. $S_n = \Pi_1^n[arb]$ is a complete meet-semilattice

   (but not a lattice: $(\forall x\ \mathbf{a}x) \vee (\forall x\ \mathbf{b}x) \equiv a^+ \cup b^+$, but $\forall x\ (\mathbf{a}x \vee \mathbf{b}x) \equiv \{a, b\}^+$)
3. $\bigcup_{n \geq 1} \Pi_1^n[arb]$ is a lattice (e.g. $(\forall x\ \mathbf{a}x) \vee (\forall x\ \mathbf{b}x) \equiv \forall x\ \forall y\ (\mathbf{a}x \vee \mathbf{b}y)$)
4. Thus, the embedding $g_n : \Pi_1^n[arb] \hookrightarrow \mathcal{P}(A^+)$ has a lower adjoint f_n.
5. We compute f_n explicitly, and we obtain $f_n[Reg] \subseteq \Pi_1^n[Reg]$.
6. Using the corollary, we conclude the desired equality. □

- Can we use the above results to get advances on *Straubing's Conjecture*?
- Many *variations of Straubing's Conjecture* exist (see e.g. McKenzie, Thomas, Vollmer, *Extensional uniformity for boolean circuits*). Will this approach work?
- It is also known that $\mathbf{FO}[\mathit{arb}] \cap \mathit{Reg} = \mathbf{FO}[\mathit{Reg}]$. The proof is difficult and involves *Boolean circuit complexity and Semigroup theory*. Can we get a simpler one?

Thank you!