A Sahlqvist theorem for subordination algebras

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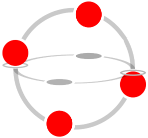
TACL 2019 - June 2019



Sahlqvist theorem

Example

For a Kripke frame (X, R) we have

$$(X, R) \models \Box p \to \Box\Box p \text{ iff } (X, R) \models x\, R\, y \wedge y\, R\, z \to x\, R\, z.$$

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Theorem

If φ is a Sahlqvist formula φ then there exists a first order formula $\alpha(\varphi)$ (in the language of the accessibility relation) such that, for a Kripke frame (X, R),

$$(X, R) \models \varphi \textit{ iff } (X, R) \models \alpha(\varphi).$$

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Example

$p \to \Diamond p$ (reflexivity), $p \to \Box\Diamond p$ (symmetry), $\Box p \to \Diamond p$ (right unboundness), ...

Subordination algebras

Definition

A subordination algebra is a pair $(B, \prec)$ where B is a Boolean algebra and $\prec$ a binary relation on B such that :

- $0 \prec 0$ and $1 \prec 1$,
- $a \prec b, c$ implies $a \prec b \wedge c$,
- $a, b \prec c$ implies $a \vee b \prec c$,
- $a \leq b \prec c \leq d$ implies $a \prec d$.

Subordination algebras as generalisation of modal algebras

Definition (Option 1)

Let $(B, \Diamond)$ be a modal algebra. Define on B the relation

$$a \prec_\Diamond b \text{ iff } \Diamond a \leq b.$$

Then, $(B, \prec_\Diamond)$ is a subordination algebra.

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Definition (Option 2)

Let $(B, \blacklozenge)$ be a modal algebra. Define on B the relation

$$a \prec_{\blacklozenge} b \text{ iff } a \leq \blacksquare b.$$

Then $(B, \prec_{\blacklozenge})$ is a subordination algebra.

Subordination morphisms

Definition

Let B, C be subordination algebras and $h : B \longrightarrow C$ a Boolean morphism. Consider the following axioms :

(w) $a \prec b$ implies $h(a) \prec h(b)$,

($\Diamond$) $h(a) \prec c$ implies $a \prec b$ and $h(b) \leq c$ for some b,

($\blacklozenge$) $a \prec h(c)$ implies $b \prec c$ and $a \leq h(b)$ for some b.

Subordination algebras as generalisation of modal algebras

Proposition

If $h : (B, \Diamond) \longrightarrow (C, \Diamond)$ *is a modal morphism, then* $h : (B, \prec_{\Diamond}) \longrightarrow (C, \prec_{\Diamond})$ *is a Boolean morphism verifying (w) and* $(\Diamond)$*.*

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Let X, Y be subordination spaces and $f : X \longrightarrow C$ a continuous function. Consider the following axioms :

(w) $x \, R \, y$ implies $f(x) \, R \, f(y)$,

($\Diamond$) $f(x) \, R \, y$ implies $x \, R \, z$ and $f(z) = y$ for some z,

($\blacklozenge$) $x \, R \, f(y)$ implies $z \, R \, y$ and $f(z) = x$ for some z.

Dual of a subordination algebra

Let $(B, \prec)$ be a subordination algebra. We denote

1. $X_B = \mathrm{Ult}(B)$ the Stone dual of B, that is the set of ultrafilters of B equipped with the topology generated by the set

$$\eta(a) = \{x \in \mathrm{Ult}(B) \mid x \ni a\},$$

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2. $R_\prec$ the binary relation on X_B defined by

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The pair $(X_B, R_\prec)$ forms a subordination space.

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The pair $(B_X, \prec_R)$ is a subordination algebra.

Duals of morphisms

Proposition

1. *If* $h : B \longrightarrow C$ *is a Boolean morphism verifying (w) (resp. (*$\Diamond$*) and (*$\blacklozenge$*)) then*

$$\mathrm{Ult}(h) : \mathrm{Ult}(C) \longrightarrow \mathrm{Ult}(B) : x \longmapsto h^{-1}(x)$$

is a continuous function that verifies (w) (resp. ($\Diamond$*) and (*$\blacklozenge$*))*

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2. *If* $f : X \longrightarrow Y$ *is a continuous function verifying (w) (resp.* $(\Diamond)$ *and* $(\blacklozenge)$*) then*

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is a Boolean morphism verifying (w) (resp. $(\Diamond)$ *and* $(\blacklozenge)$*).*

Duality

Theorem

1. *If* $(B, \prec)$ *is a subordination algebra then*

$$\eta : (B, \prec) \longrightarrow (\mathrm{Clop}(\mathrm{Ult}(B)), \prec_{R_\prec}) : a \longmapsto \{x \in \mathrm{Ult}(B) \mid x \ni a\}$$

is a bijective Boolean morphism that verifies (w), ($\Diamond$) and ($\blacklozenge$) and such that

$$\eta(a) \prec \eta(b) \Rightarrow a \prec b.$$

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2. *If* (X, R) *is a subordination space then*

$$\varepsilon : (X, R) \longrightarrow (\mathrm{Ult}(\mathrm{Clop}(X)), R_{\prec_R}) : x \longmapsto \{O \in \mathrm{Clop}(X) \mid O \ni x\}$$

is a bijective continuous function that verifies (w), $(\Diamond)$ *and* $(\blacklozenge)$ *and such that*

$$\varepsilon(x) \, R \, \varepsilon(y) \Rightarrow x \, R \, y.$$

Validity for subordination spaces

Definition

Let (X, R) be a subordination space. A valuation on X is a map $v : \mathrm{Var} \longrightarrow \mathrm{Clop}(X)$. The valuation is then extend to bimodal formulas in the usual way :

$$v(\Diamond\psi) = R(-, v(\psi)) \qquad v(\Box\psi) = R(-, v(\psi)^c)^c$$
$$v(\blacklozenge\psi) = R(v(\psi), -) \qquad v(\blacksquare\psi) = R(v(\psi)^c, -)^c$$
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$$\ldots$$

A bimodal formula φ is valid in X for the valuation v, denoted by $X \models_v \varphi$, if $v(\varphi) = X$.

Remark on valuation

Remark

Since the access relation of X is solely closed, the valuation of a bimodal formula may fail to be clopen.
For instance,

$$v(\Diamond p) = R(-, v(p)) = \{x \mid \exists y \in v(p) : x \; R \; y\}$$

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In particular, this means that we cannot extend a valuation on a subordination algebra $v : \mathrm{Var} \longrightarrow B$ to all bimodal formulas.

Canonical extension of a subordination algebra

Let B be a subordination algebra. Then $\mathrm{Ult}(B)$ is a subordination algebra, and so, in particular, a Kripke frame.

It follows that $B^\delta = \mathcal{P}(\mathrm{Ult}(B))$ is a complete tense bimodal algebra with for every $E \in \mathcal{P}(\mathrm{Ult}(B))$

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Proposition

The map

$$\eta : B \longrightarrow B^\delta : a \longmapsto \{x \in \mathrm{Ult}(B) \mid x \ni a\}$$

is an injective Boolean morphism such that

$$a \prec b \Leftrightarrow \Diamond\eta(a) \leq \eta(b) \Leftrightarrow \eta(a) \leq \blacksquare\eta(b).$$

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Let $(B, \prec)$ be a subordination algebra. A valuation on B is a map $v : \mathrm{Var} \longrightarrow B$. A valuation on B can be extended to a valuation $\eta \circ v : \mathrm{Var} \longrightarrow B^\delta$ on B^δ.

[1] Remember that B and B^δ share the same top element.

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A bimodal formula φ is valid in B under the valuation v if $\eta(v(\varphi)) = 1$[1], that is φ is valid in B^δ under the valuation $\eta \circ v$.

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Proposition

Let φ be a bimodal formula. We have $B \models \varphi$ if and only if $\mathrm{Ult}(B) \models \varphi$.

[1] Remember that B and B^δ share the same top element.

Translations

There are several correspondence problems that could be studied.

1. Translation of bimodal formulas on a subordination algebra into first order formulas of the accessibility relation of the dual.

$$(B, \prec) \models \blacksquare p \to \blacklozenge p \text{ iff } (X, R) \models (\forall x)(\exists y)(y \; R \; x).$$

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2. Translation of first order properties in the language of subordination algebras into first order formulas of the accessibility relation of the dual.

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s-positive formulas

Definition

1. A bimodal formula is closed (resp. open) if it is obtained from constants $\top$, $\bot$, propositional variables and their negations by applying only $\wedge$, $\vee$, $\Diamond$ and $\blacklozenge$ (resp. $\wedge$, $\vee$, $\Box$ and $\blacksquare$).

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Remark

A *s*-positive formula is a positive formula where no $\Box$ or $\blacksquare$ is under the scope of a $\Diamond$ or a $\blacklozenge$ and vice-versa for a *s*-negative formula. (Restriction needed for the intersection lemma to work)

Sahlqvist formulas

Definition

1. A strongly positive bimodal formula is a conjunction of formulas of the form
$$\Box^{\langle \mu \rangle} p = \Box^{\mu_1} \blacksquare^{\mu_2} \ldots \Box^{\mu_n} p$$
where $p \in \mathrm{Var}$, $n \in \mathbb{N}$ and $\mu \in \mathbb{N}^n$.

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3. A s-Sahlqvist formula is a formula of the form
$$\Box^{\langle \mu \rangle} (\varphi_1 \to \varphi_2)$$
where φ_1 is s-untied and φ_2 is s-positive.

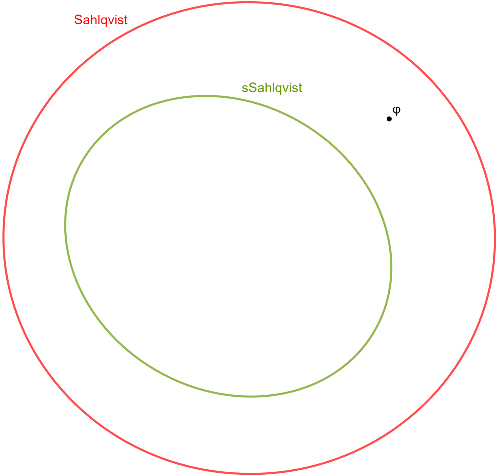
Sahlqvist theorem

Theorem

Let φ be a s-Sahlqvist formula. There exists a first order formula $\alpha(\varphi)$ in the language of a binary relation such that for any subordination algebra $(B, \prec)$ with dual (X, R) we have

$$(B, \prec) \models \varphi \text{ iff } (X, R) \models \alpha(\varphi).$$

Existence of a counterexample ?



A remark on substitution

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Then (X, R) is a subordination space such that

$$(X, R) \models p \to \Diamond\Box p.$$

But for $\varphi = p \wedge \neg\Box p$, we have that

$$(X, R) \not\models \varphi \to \Diamond\Box\varphi.$$

Scheme-extensible formulas

Definition

A bimodal formula φ is said to be scheme-extensible if $\underline{B} \models \varphi(\overline{p})$ (we write $\varphi(\overline{p})$ to indicate that the variables of φ are among the tuple $\overline{p}$) implies $\underline{B} \models \varphi(\overline{\psi})$ for all $\overline{\psi}$.

Theorem

Any s-Sahlqvist bimodal formula is scheme-extensible.

Counterexample

The formula $\varphi \equiv p \longrightarrow \Diamond\Box p$ is a Sahlqvist formula (for modal algebras), corresponding to

$$(\forall x)(\exists y)(x \, R \, y \text{ and } R(y, -) \subseteq \{x\},$$

but φ is not scheme-extensible and hence φ is not a s-Sahlqvist formula.

Sloth

Post credits slide

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Post credits slide

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Question : How can we deduct $p \to \Box\Diamond p$ from $p \to \Diamond\Box p$.