Artin Glueings as semidirect products

Peter Faul
peter@faul.io
University of Cambridge
(Joint work with Graham Manuell)

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Artin glueings of topological spaces

Let $N = (|N|, \mathcal{O}(N))$ and $H = (|H|, \mathcal{O}(H))$ be topological spaces.

What topological spaces $G = (|G|, \mathcal{O}(G))$ satisfy that H is an open subspace and N its closed complement?

Such a space G we call an Artin glueing of H by N.

It must be that $|G| = |N| \sqcup |H|$.

Each open $U \in \mathcal{O}(G)$ then corresponds to a pair (U_N, U_H) where $U_N = U \cap N$ and $U_H = U \cap H$.

Thus $\mathcal{O}(G)$ is isomorphic to the frame L_G of certain pairs (U_N, U_H) with componentwise union and intersection.

The associated meet-preserving map

For each $U \in \mathcal{O}(H)$ there is a largest open $V \in \mathcal{O}(N)$ such that (V, U) occurs in L_G.

Let $f_G \colon \mathcal{O}(H) \to \mathcal{O}(N)$ be a function which assigns to each $U \in \mathcal{O}(H)$ the largest V.

This function preserves finite meets.

Artin glueings from meet-preserving maps

The space G can be recovered entirely from f_G via $(V, U) \in L_G$ if and only if $V \subseteq f_G(U)$

- Assume $V \subseteq f_G(U)$.
- For any $V \in \mathcal{O}(N)$ there exists a pair $(V, U_V) \in L_G$ and for any $U \in \mathcal{O}(H)$ there is a pair $(\emptyset, U) \in L_G$.
- Now simply consider the union of $(\emptyset, U)$ with $(V, U_V) \cap (f_G(U), U) = (V, U_V \cap U)$, which results in (V, U).

Applying the above process to any meet preserving map $f \colon \mathcal{O}(H) \to \mathcal{O}(N)$ will give a space $\mathrm{Gl}(f)$ satisfying our requirements.

Thus Artin glueings are given by finite-meet preserving maps f and we call $\mathrm{Gl}(f)$ the Artin glueing along f.

Semidirect products of groups are analogous

Let N and H be groups.

A semidirect product of H by N is a group G satisfying the following conditions.

1. H is a subgroup and N a normal subgroup.
2. $N \cap H = \{e\}$.
3. $N \vee H = NH = \{nh : n \in N, h \in H\} = G$.

Conditions 2 and 3 together say that N and H are complements.

Every such group G is determined by a unique group homomorphism $\alpha\colon H \to \mathrm{Aut}(N)$.

Split extensions of groups

A diagram $N \rightarrowtail^{k} G \underset{s}{\overset{e}{\rightleftarrows}} H$ is a split extension when the following hold:

1. k is the kernel of e,
2. e is the cokernel of k,
3. $se = \mathrm{id}$.

Every semidirect product of N and H yields a split extension.

Every split extension is of this form.

Every element $g \in G$ can be written $g = k(n)s(h)$ for $n \in N$ and $h \in H$. Thus G is generated by the images of k and s.

Split extensions in a category

Kernels and cokernels require zero-morphisms $0_{X,Y} \colon X \to Y$ for all combinations of objects X and Y.

A class of zero-morphisms in a category satisfies that for any morphisms $f \colon W \to X$ and $g \colon Y \to Z$ we have $0_{X,Y} f = 0_{W,Y}$ and $g 0_{X,Y} = 0_{X,Z}$.

Now we can define kernels and cokernels.

1. The kernel of a map $f \colon X \to Y$ is the equaliser of f and $0_{X,Y}$.
2. The cokernel of a map $f \colon X \to Y$ is the coequaliser of f and $0_{X,Y}$

Thus we can now consider split extensions in any pointed category.

The appropriate category of frames

We want a pointed category in which the generating extensions of H by N are precisely the Artin glueings of N and H.

The usual category of frames is no good as it does not have zero-morphisms.

Instead consider the category RFrm whose objects are frames and whose morphisms are finite-meet preserving maps.

The maps $\top_{X,Y} \colon X \to Y$ sending each element of X to the top element 1 of Y form a class of zero morphisms.

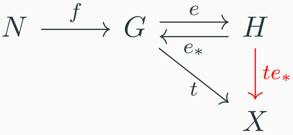
Cokernels

The cokernel of a morphism $f\colon N \to G$ in RFrm exists and is $e\colon G \to {\downarrow} f(0)$, where $e(g) = g \wedge f(0)$.

We call such a map a normal epimorphism.

Furthermore e has a right adjoint section $e_*(x) = (f(0) \Rightarrow x)$.

- Let $t\colon G \to X$ be such that $tf = \top$.



- If $e(x) = e(y)$ (i.e. $x \wedge f(0) = y \wedge f(0)$) then

  $$t(x) = t(x) \wedge 1 = t(x) \wedge t(f(0)) = t(x \wedge f(0)) = t(y \wedge f(0)) = t(y)$$

- Because $e(e_* e(x)) = e(x)$ we have that $t(e_* e(x)) = t(x)$.
- Thus $t = te_* e$ and te_* is the unique such map as e is epic.

Kernels

Kernels do not always exist in RFrm, but kernels of normal epis do.

The kernel of a normal epi $e\colon G \to {\downarrow}u$ is the inclusion

$$k\colon {\uparrow}u \hookrightarrow G.$$

It has a left adjoint $k^*(x) = x \vee u$.

Thus split extensions in RFrm are of the form

$$\uparrow u \xrightarrow{\;k\;} G \underset{s}{\overset{e}{\rightleftarrows}} \downarrow u$$

Notice that ${\downarrow}u$ is an open sublocale of G and ${\uparrow}u$ is a closed sublocale.

Generating extensions and the splitting

Recall that if $N \overset{k}{\rightarrowtriangle} G \underset{s}{\overset{e}{\rightleftarrows}} H$ is a split extension of groups, then each element of g can be written $g = k(n)s(h)$.

We call $N \overset{k}{\rightarrowtriangle} G \underset{s}{\overset{e}{\rightleftarrows}} H$ a generating extension in RFrm if each element of $g \in G$ can be written $g = k(n) \wedge s(h)$.

A split extension will be generating if and only if s is the right adjoint of e_*.

Thus a generating extension is entirely determined by a normal epi!

Artin glueings are generating extensions

Recall that the Artin glueing $\mathrm{Gl}(f)$ of a finite-meet preserving map $f\colon H \to N$ is the frame of pairs (n, h) where $n \leq f(h)$.

We have projections $\pi_1\colon \mathrm{Gl}(f) \to N$ and $\pi_2\colon \mathrm{Gl}(f) \to H$ which preserve finite meets.

These projections have right adjoints which lie in RFrm.

1. $\pi_{1*}(n) = (n, 1)$.
2. $\pi_{2*}(h) = (f(h), h)$.

We have that $N \overset{\pi_{1*}}{\rightarrowtail} \mathrm{Gl}(f) \underset{\pi_{2*}}{\overset{\pi_2}{\rightleftarrows}} H$ is a generating extension.

Think of π_2 as the map $(-) \wedge (0,1)\colon \mathrm{Gl}(f) \to {\downarrow}(0,1)$

Notice that f can be recovered via $f = \pi_1\pi_{2*}$.

All extensions are Artin glueings

From a generating extension $N \overset{k}{\rightarrowtail} G \underset{e_*}{\overset{e}{\rightleftarrows}} H$ we can form

$$N \overset{\pi_{1*}}{\rightarrowtail} \mathrm{Gl}(k^* e_*) \underset{\pi_{2*}}{\overset{\pi_2}{\rightleftarrows}} H.$$

The frames G and $\mathrm{Gl}(k^* e_*)$ are isomorphic

1. $f \colon G \to \mathrm{Gl}(k^* e_*)$ sends g to $(k^*(g), e(g))$,
2. $f^{-1} \colon \mathrm{Gl}(k^* e_*) \to G$ sends (n, h) to $k(n) \wedge e_*(h)$.

Furthermore f and f^{-1} make the diagram below commute.

$$\begin{array}{ccccc}
N & \xrightarrow{\quad k \quad} & G & \underset{e_*}{\overset{e}{\rightleftarrows}} & H \\
\Vert & & f^{-1}\big\uparrow\big\downarrow f & & \Vert \\
N & \xrightarrow{\pi_{1*}} & \mathrm{Gl}(k^* e_*) & \underset{\pi_{1*}}{\overset{\pi_1}{\rightleftarrows}} & H
\end{array}$$

Further thoughts

Since Artin glueings correspond to finite-meet preserving maps, we have that $\mathrm{Hom}(-,-)$ is the bifunctor of generating extensions.

The homsets of RFrm are actually meet-semilattices. This operation gives a Baer sum on generating extensions.

These ideas are related to $\mathcal{S}$-protomodularity and Schreier extensions.

The Artin glueing construction works on toposes and there is an analogue on that level.