Duality for b-frames and complete lattices

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Plan

Motivation

- Aim of the talk:
  - Introduce the category **bFrm**, dual to the category **cLat** of complete lattices and complete lattice homomorphisms.
  - Present some preliminary results regarding a correspondence theory between lattice equations and b-frame properties. I will mostly focus on certain classes of Heyting algebras.
- Motivation: Groundwork for investigating Kripke, topological and locale completeness of intermediate logics.

Background

1. In the forcing literature: study of properties of cBA via representations as regular opens of posets, and study of forcing posets via their Boolean completions...

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2. Esakia duality as a framework for investigating Kripke completeness and canonicity for intermediate logics;

3. Dualities between bi-topological spaces and lattices: Hartonas[1997] and Allwein[2001] dualities for bounded lattices, d-frames (Jung & Moshier [2006]), choice-free Stone duality (Bezhanishvili & Holliday[2019])...

Plan

The category **bFrm***

- A *b-frame* is a bi-preordered set $\mathcal{X} = (X, \leq_1^X, \leq_2^X)$.
- A *b-morphism* $f : \mathcal{X} \to \mathcal{Y}$ is a map satisfying the following properties:
  1. (Monotonicity) $x \leq_i^X x' \Rightarrow f(x) \leq_i^Y f(x')$ for any $x, x' \in X$, $i \in \{1, 2\}$;
  2. (1-covering) for any $x \in X$, $y \in Y$ such that $f(x) \leq_2^Y y$, there is $x' \geq_2^X x$ such that $f(x') \geq_1^Y y$;
  3. (2-covering) for any $x \in X$, $y \in Y$ such that $f(x) \leq_1^Y y$, there is $x' \geq_1^X x$ such that $f(x') \geq_2^Y y$.
- **bFrm*** is the category of b-frames and b-morphisms between them.

Special cases of b-frames

Any poset $(P, \leq)$ can be regarded as a b-frame in two distinct ways:

- As a *Kripke* b-frame $(P, \leq, \Delta_P)$; b-morphisms $f : \mathcal{X} \to \mathcal{Y}$ between Kripke b-frames coincide with *p-morphisms* (i.e. $\uparrow f(x) = f[\uparrow x]$ for any $x \in X$);
- As a *Boolean* b-frame $(P, \leq, \leq)$; b-morphisms $f : \mathcal{X} \to \mathcal{Y}$ between Boolean b-frames satisfy a weaker condition than p-morphisms, namely that $\uparrow f(x)$ is *dense* in $f[\uparrow x]$ for any $x \in X$.

Independence

Definition

Let $\mathcal{X} = (X, \leq_1^X, \leq_2^X)$ a b-frame, and $x, y \in X$. We say that y is *independent* from x (noted $x_2 \perp_1 y$) if there exists no $z \in X$ such that $x \leq_2 z$ and $y \leq_1 x$.

Remark

- For Kripke b-frames, y is independent from x iff $y \not\leq_1 x$.
- For Boolean b-frames, independence boils down to incompatiblity.

The functor ρ

- Given a b-frame $\mathcal{X} = (X, \leq_1, \leq_2)$, let $I_1 : \mathscr{C}_2 \to \mathscr{O}_1$ and $C_2 : \mathscr{C}_2 \to \mathscr{O}_1$ be the interior and closure operators corresponding to the upset topologies induced by $\leq_1$ and $\leq_2$ respectively.

- $I_1 C_2 : \mathscr{O}_1 \to \mathscr{O}_1$ is a closure operator. Its fixpoints are called the *generalized regular opens* (noted $\mathrm{RO}_{12}(\mathcal{X})$), and form a complete lattice.

- This can be seen a generalization of Tarski's result that the regular opens of any topological space form a *cBA*.

The functor ρ

Lemma

Let $f : \mathcal{X} \to \mathcal{Y}$ be a b-morphism. Then $f^{-1} : \mathrm{RO}_{12}(\mathcal{Y}) \to \mathrm{RO}_{12}(\mathcal{X})$ is a complete lattice homomorphism.

Definition

The contravariant *regular open functor* $\rho : \mathbf{bFrm}^* \to \mathbf{cLat}$ is defined as follows:

- $\rho(\mathcal{X}) = \mathrm{RO}_{12}(\mathcal{X})$ on objects;
- $\rho(f) = f^{-1} : \mathrm{RO}_{12}(\mathcal{Y}) \to \mathrm{RO}_{12}(\mathcal{X})$ for a b-morphism $f : \mathcal{X} \to \mathcal{Y}$.

From lattices to b-frames

Definition

Let L be a complete lattice. The *dual Allwein b-frame* of L, noted $\alpha(L) = (P_L, \leq_1^L, \leq_2^L)$ is defined as follows:

- $P_L = \{(a, b) \in L \times L \ ; \ a \not\leq_L b\}$;
- $(a_1, b_1) \leq_1 (a_2, b_2)$ iff $a_2 \leq_L a_1$;
- $(a_1, b_1) \leq_2 (a_2, b_2)$ iff $b_1 \leq_L b_2$;

Definition

Let $f : L \to M$ be a complete lattice homomorphism. Since f preserves all limits and all colimits, f has left and right adjoints $-^f, -_f : M \to L$. Let $\alpha(f) : \alpha(M) \to \alpha(L)$ be defined as $\alpha(f)(a, b) = (a^f, b_f)$.

From lattices to b-frames

Lemma

Let $f : L \to M$ *be a complete lattice homomorphism. Then* $\alpha(f) : \alpha(M) \to \alpha(L)$ *is a b-morphism.*

Definition

The contravariant *Allwein functor* $\alpha : \mathbf{cLat} \to \mathbf{bFrm}^*$ is defined as follows:

- $\alpha(L)$ is the dual Allwein b-frame of L on objects;
- $\alpha(f) : \alpha(M) \to \alpha(L)$ is defined as before for any complete lattice homomorphism $f : L \to M$.

An adjunction between $\mathbf{bFrm}^*$ and $\mathbf{cLat}$

Theorem

The functors α and ρ form a contravariant adjunction:

$$\mathbf{cLat}^{op} \underset{\alpha}{\overset{\rho}{\rightleftarrows}} \mathbf{bFrm}^* \qquad (\bot)$$

- *For any complete lattice L, the counit $\epsilon_L : L \to \rho\alpha(L)$ is given by $\epsilon_L(a) = \uparrow_1(a, 0)$.*
- *For any b-frame $\mathcal{X}$, the unit $\eta_{\mathcal{X}} : X \to \alpha\rho(\mathcal{X})$ is given by $\eta_{\mathcal{X}}(x) = (U^x, V_x)$, where $U^x = I_1 C_2(\uparrow_1 x)$ and $V_x = \{y \in X \,;\, x_2 \bot_1 y\}$.*

Dense embeddings

Definition

Let $f : \mathcal{X} \to \mathcal{Y}$ a b-morphism.

- f is an *embedding* if for any $x, y \in X$, $x_2\perp_1 y$ iff $f(x)_2\perp_1 f(y)$.
- f is *dense* if for any $y \in Y$, there is $x \in X$ such that $f(x) \geq_{12}^{Y} y$.

Remark

- Dense embeddings between Kripke b-frames correspond to bijective p-morphisms.
- Dense embeddings between Boolean b-frames correspond to forcing equivalence for posets.

Dense embeddings

Lemma

- *For any* $L, M \in \mathbf{cLat}$ *and* $f \in \mathbf{Hom}(L, M)$*:*
  - f *is injective iff* $\alpha(f)$ *is dense;*
  - f *is surjective iff* $\alpha(f)$ *is an embedding.*
- *For any* $\mathcal{X}, \mathcal{Y} \in \mathbf{bFrm}^*$ *and* $f \in \mathbf{Hom}(\mathcal{X}, \mathcal{Y})$*:*
  - f *is an embedding iff* $\rho(f)$ *is surjective;*
  - f *is dense iff* $\rho(f)$ *is injective.*

Corollary (Allwein)

For any complete lattice L*,* $\epsilon_L : L \to \rho\alpha(L)$ *is an isomorphism. As a consequence,* α *is fully faithful, and* $\rho \dashv \alpha$ *is an idempotent adjunction.*

Proof.

The identity map on $\alpha(L)$ is a dense embedding. □

Towards a duality

- Conversely, for any b-frame $\mathcal{X}$, $\eta_{\mathcal{X}} : \mathcal{X} \to \alpha\rho(\mathcal{X})$ is a dense embedding. In general however, it will not be an isomorphism (i.e. a bijective b-morphism reflecting both orders).
- Question: Can we characterize those b-frames for which $\eta_{\mathcal{X}}$ is an isomorphism, or equivalently, the range of the functor α?

Normal b-frames

Definition

Let $\mathcal{X} = (X, \leq_1^X, \leq_2^X)$ be a b-frame.

- $\mathcal{X}$ is *antisymmetric* if $\leq_1^X \cap \leq_2^X$ is a partial order.
- $\mathcal{X}$ is 1-*separative* iff for any $x, y \in X$, $x \leq_1^X y \Leftrightarrow \forall z(z_2 \bot_1 x \rightarrow z_2 \bot_1 y)$.
- $\mathcal{X}$ is 2-*separative* iff for any $x, y \in X$, $x \leq_2^X y \Leftrightarrow \forall z(x_2 \bot_1 z \rightarrow y_2 \bot_1 z)$.
- $\mathcal{X}$ *has enough points* iff for any $U \nsubseteq V \in RO_{12}(\mathcal{X})$, there is $z \in \mathcal{X}$ such that $U = U^z$ and $V = V_z$.

A b-frame is *separative* if it is antisymmetric and 1&2-separative.
It is *normal* if it is separative and has enough points.

Normal b-frames

Theorem

Let $\mathcal{X}$ be a b-frame. The following are equivalent:

1. $\mathcal{X}$ *is a normal b-frame;*
2. $\eta_{\mathcal{X}} : \mathcal{X} \to \alpha\rho(\mathcal{X})$ *is an isomorphism.*

Corollary

Every b-frame densely embeds into a 1&2*-separative b-frame.*

- This can be seen as a generalization of the well known result that any poset is forcing equivalent to a separative poset.

Duality

Definition

bFrm is the category of normal b-frames and b-morphisms between them.

Theorem

The functors α and ρ restrict to a dual equivalence between **cLat** *and* **bFrm***:*

$$\alpha : \mathbf{cLat}^{op} \cong \mathbf{bFrm} : \rho$$

Plan

Correspondence theory

- Template: correspondence between equations on modal or Heyting algebras and first-order conditions on Kripke frames.
- The goal is to match equations valid in a class $\mathcal{K}$ of lattices with conditions, preferably first-order, which hold precisely on those b-frames whose dual lattices belong to $\mathcal{K}$.
- In general, it is possible to find conditions on b-frames that are sufficient for the dual lattice to belong to $\mathcal{K}$. But for necessary and sufficient conditions, it is preferable to focus first on normal b-frames instead of arbitrary b-frames.

Heyting algebras

- Example: Given a b-frame $\mathcal{X}$, if $I_1 C_2$ is a *nucleus* on $\mathscr{O}_1$ (i.e. preserves finite meets), then $\mathrm{RO}_{12}(\mathcal{X})$ is a *cHA*.
- On the other hand, for any b-frame $\mathcal{X} = (X, \leq_1^X, \leq_2^X)$ such that $\leq_1^X = \Delta_X$, $\mathrm{RO}_{12}(\mathcal{X})$ is a *cHA*, but in general $I_1 C_2$ is not a nucleus.
- However, the two conditions coincide for *normal* b-frames.

Heyting algebras

Definition

Let $\mathcal{X} = \{X, \leq_1^X, \leq_2^X)$ be a b-frame.

- For any $x, y \in X$, let $x_{12}\bot_1 y$ iff there is no $z \in X$ such that $x \leq_{12} z$ and $y \leq_1 z$.
- A *Heyting point* in $\mathcal{X}$ is a point $x \in X$ such that for any $y \in X$, $x_2\bot_1 y \Leftrightarrow x_{12}\bot_1 y$. The set of Heyting points of $\mathcal{X}$ is denoted $H(\mathcal{X})$.
- A b-frame is *Heyting* if $H(\mathcal{X})$ is dense in $\mathcal{X}$.

Lemma

Let $\mathcal{X} = \{X, \leq_1^X, \leq_2^X)$ be a b-frame. If $\mathcal{X}$ is a Heyting b-frame, then $I_1 C_2$ is a nucleus.

Heyting algebras

Lemma

If $\mathcal{X} = \alpha(L)$ for some cHA L, then $H(\mathcal{X})$ is dense in $\mathcal{X}$.

Proof.

Any point in $\alpha(L)$ is of the form (a, b), for some $a \not\leq b \in L$. But then $(a, a \to b)$ is a Heyting point in $\alpha(L)$ which is a 12-successor of (a, b). □

Corollary

1. *A normal b-frame $\mathcal{X}$ is Heyting iff $\rho(\mathcal{X})$ is a cHA.*
2. *For any b-frame $\mathcal{X}$, $\rho(\mathcal{X})$ is a cHA iff $\mathcal{X}$ densely embeds into a Heyting b-frame.*

Heyting algebras

Definition

A *h-morphism* is a b-morphism $f : \mathcal{X} \to \mathcal{Y}$ between two Heyting b-frames which satisfies the following extra condition:

- (12-covering) for any $x \in X$, $y \in Y$ such that $f(x) \leq_1^Y y$, there is $x' \geq_1^X$ such that $f(x') \geq_{12}^Y y$.

Lemma

- *For any Heyting homomorphism* $f : L \to M$, $\alpha(f) : \alpha(M) \to \alpha(L)$ *is a h-morphism.*
- *For any h-morphism* $f : \mathcal{X} \to \mathcal{Y}$, $\rho(f) : \rho(\mathcal{Y}) \to \rho(\mathcal{X})$ *is a Heyting homomorphism.*

Corollary

The category **HbFrm** *of normal Heyting b-frames and h-morphisms is dual to the category* **cHA**.

Similar results can be obtained for some other varieties, including distributive lattices, co-Heyting algebras, Boolean algebras...

Definition

- A *Boolean point* in a *b*-frame $\mathcal{X}$ is a point x such that for any $y \in X$, $x_1 \perp_{12} y$ iff $x_2 \perp_{12} y$.
- A b-frame is *Boolean* if its set of Boolean points $B(\mathcal{X})$ is dense in $\mathcal{X}$.

Lemma

The category **BbFrm** *of Boolean b-frames and h-morphisms is dual to the category* **cBA**.

Spatial and Kripke locales

Definition

- A *spatial locale* is a locale L such that $L \cong \Omega(X)$ for some topological space (X, τ).
- A *Kripke locale* is a locale L such that $L \cong Up(X)$ for some pre-ordered set $(X, \leq)$.

Every locale is isomorphic to $\mathrm{RO}_{12}(\mathcal{X})$ for some Heyting b-frame $\mathcal{X}$.

Question: Can we identify conditions on b-frames that characterize spatial and Kripke locales?

Spatial locales

Definition

Let L be a locale. A *meet-prime element* of L is an element $b \in L$ such that for any $c, d \in L$, $b \leq c \wedge d \Rightarrow b \leq c$ or $b \leq d$.

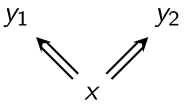
Theorem (Folklore)

A locale L is spatial if and only if for any $a, b \in L$ such that $a \nleq b$, there is a meet-prime $c \in L$ such that $a \nleq c$ and $b \leq c$.

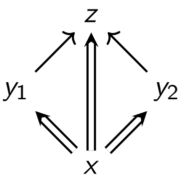
Spatial locales

Definition

Let $\mathcal{X}$ be a Heyting b-frame. A point $x \in X$ is *meet-prime* if for any $y_1, y_2 \in X$, every diagram of the form:



can be completed as follows:



The set of meet-prime points of $\mathcal{X}$ is noted $M(\mathcal{X})$.

Spatial locales

Lemma

- *Let $\mathcal{X}$ be a Heyting b-frame such that $M(\mathcal{X})$ is dense in $\mathcal{X}$. Then $\rho(\mathcal{X})$ is spatial.*
- *For any spatial locale L, $M(\alpha(L))$ is dense in $\alpha(L)$.*

Corollary

For any normal Heyting b-frame $\mathcal{X}$, $\rho(\mathcal{X})$ is spatial iff $M(\mathcal{X})$ is dense in $\mathcal{X}$.

Kripke locales

Definition

Let L be a locale. A *splitting pair* in L is a pair $(a, b) \in L \times L$ such that $a \not\leq b$ and for any $c \in L$, $a \leq c$ or $c \leq b$.

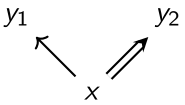
Theorem

A locale L is Kripke if and only if for any pair of elements (a, b) in $L \times L$ such that $a \not\leq b$, there is a splitting pair (a', b') such that $a' \leq a$ and $b \leq b'$.

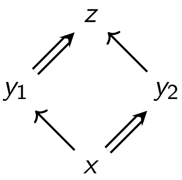
Kripke locales

Definition

Let $\mathcal{X}$ be a Heyting b-frame. A point $x \in X$ is *splitting* if for any $y_1, y_2 \in X$, every diagram of the form:



can be completed as follows:



The set of splitting points of $\mathcal{X}$ is noted $S(\mathcal{X})$.

Kripke locales

Lemma

- *Let $\mathcal{X}$ be a Heyting b-frame such that $S(\mathcal{X})$ is dense in $\mathcal{X}$. Then $\rho(\mathcal{X})$ is Kripke.*
- *For any Kripke locale L, $S(\alpha(L))$ is dense in $\alpha(L)$.*

Corollary

For any normal Heyting b-frame $\mathcal{X}$, $\rho(\mathcal{X})$ is Kripke iff $S(\mathcal{X})$ is dense in $\mathcal{X}$.

Plan

Next steps

- Transfer techniques from standard Kripke semantics to the larger setting of b-frame semantics for IPC. Goal: Make some progress towards a solution to Kuznetsov's problem.
- Extend the correspondence theory to more varieties: modular lattices, ortholattices, BAO's... Goal: Develop an analogue to Sahlqvist theory in the setting of lattice equations and b-frames.
- Explore connections with filter-ideal based dualities. Goal: Understand how b-frames fit in the larger picture of Stone-like dualities.

Thank You!