Nearness Posets

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Classical Uniformities

- Familiar concepts from metric spaces, e.g. uniform continuity, completion etc., can be generalised to uniform spaces.
- Following Tukey (1940), a uniformity is just a family of 'uniform' covers on a set X satisfying certain conditions.
- These covers determine a canonical topology, specifically where the 'stars' of $x \in X$ form a neighbourhood base at x.

Theorem

A topological space X is uniformisable iff X is completely regular.

- To extend to regular and even T_1 spaces, Morita (1951) weakened the star-refinement axiom for the covers.
- Katětov (1963) and Herrlich (1974) independently came up with equivalent versions of Morita's generalised uniformities.
- These are now usually called nearness spaces, i.e. a nearness is again just a special family of 'uniform' covers of a set X.

Nearness Frames

- More recently, people have considered point-free nearnesses.
- First, take a frame $\mathbb{L}$, i.e. a complete lattice which we think of as representing the open sets of a topological space.
- Again, a nearness is a family of covers, which are now subsets $C \subseteq \mathbb{L}$ each with $\bigvee C = 1$, again subject to some conditions.
- However, we can already see a key difference between this approach and the classical pointy notion of a nearness – Now the nearness is placed on top of a pre-existing topological structure, rather than defining it like in the classical case.

Question

What if we instead replace the lattice structure with covers?

- So we would instead start with just a set S together with some distinguished family of subsets $\Theta \subseteq \mathcal{P}(S)$, nothing more.
- Here we could even consider S to represent a more general basis or even just a subbasis of open sets of some space.

Recovering Spaces from Covers

- First question – can we recover a space X from such a weak abstract covering structure? Yes, as long as X is T_1.
- To see this, take a subbasis S of a T_1 space X with covers

$$\Theta = \{C \subseteq S : X \subseteq \bigcup S\}.$$

- For each $x \in X$, consider its subbasic neighbourhoods

$$N_x = \{s \in S : x \in s\}.$$

- As each $C \in \Theta$ covers X, each N_x is Θ-Cauchy, i.e.

$$C \in \Theta \qquad \Rightarrow \qquad N_x \cap C \neq \emptyset. \qquad\qquad (\Theta\text{-Cauchy})$$

- As X is T_1, each N_x is minimal Θ-Cauchy, i.e.

$$s \in N_x \qquad \Rightarrow \qquad \exists C \in \Theta \ (N_x \cap C = \{s\}).$$

- Moreover, there are no other minimal Θ-Cauchy subsets:
  - Say $M \subseteq S$ does not contain any N_x.
  - Then $X \subseteq \bigcup S \setminus M \in \Theta$ so M is not Θ-Cauchy.

The Spectrum

Definition

Given $\Theta \subseteq \mathcal{P}(S)$, the spectrum is the space

$$\widehat{\Theta} = \{N \subseteq S : N \text{ is minimal } \Theta\text{-Cauchy}\}$$

with the topology generated by the sets $\widehat{\Theta}_s = \{N \in \widehat{\Theta} : s \in N\}$.

- So what we just proved is the following.

Proposition

If S is a subbasis of a T_1 space X and $\Theta = \{C \subseteq S : X \subseteq \bigcup S\}$ is the family of all S-covers of X then $\widehat{\Theta}$ is homeomorphic to X.

- Conversely, say we start with abstract $\Theta \subseteq \mathcal{P}(S)$.
- By minimality, $\widehat{\Theta}$ is a T_1 space.
- By Cauchyness, each $C \in \Theta$ yields a cover $(\widehat{\Theta}_c)_{c \in C}$ of $\widehat{\Theta}$.
- However, there could be many other covers, e.g. if $C \in \Theta$ then any $D \supseteq C$ also covers $\widehat{\Theta}$, even when $D \notin \Theta$.
- Also, we could have $\widehat{\Theta}_s = \widehat{\Theta}_t$ even when $s \neq t$.

The Canonical Order

- Any $\Theta \subseteq \mathcal{P}(S)$ defines a preorder on S by

$$s \leq_\Theta t \qquad \Leftrightarrow \qquad \Theta^s \subseteq \Theta^t,$$

  where $\Theta^s = \{D \subseteq S : \{s\} \cup D \in \Theta\}$.
- If S is a concrete subbasis of some T_1 space X and Θ is the family of all covers then $\leq_\Theta$ coincides with containment, i.e.

$$s \leq_\Theta t \qquad \Leftrightarrow \qquad s \subseteq t.$$

- For abstract S and Θ, we do at least have

$$s \leq_\Theta t \qquad \Rightarrow \qquad \widehat{\Theta}_s \subseteq \widehat{\Theta}_t.$$

- So if we hope to represent S faithfully as a subbasis on the spectrum then, at the very least, $\leq_\Theta$ should be a partial order.
- In this case, let us call $(S, \leq_\Theta, \Theta)$ a nearness poset.

Finitary Nearness Posets

- Call Θ finitary if every $C \in \Theta$ is finite and, for all finite $F \subseteq S$,
$$F \supseteq C \in \Theta \qquad \Rightarrow \qquad F \in \Theta.$$
- We can now reformulate a classical result due to Wallman.

Theorem (Wallman 1938)

If $(S, \leq, \Theta)$ is a finitary nearness poset then $\widehat{\Theta}$ is compact,
$$s \leq t \qquad \Leftrightarrow \qquad \widehat{\Theta}_s \subseteq \widehat{\Theta}_t$$
$$\text{and} \qquad \Theta = \{F \subseteq S : F \text{ is finite and } \widehat{\Theta} \subseteq \bigcup_{s \in F} \widehat{\Theta}_s\}.$$

- Conversely, if S is a subbasis of compact T_1 X and Θ is the family of all finite covers, $(S, \subseteq, \Theta)$ is a finitary nearness poset.
- So we have a kind of duality
$$\text{Finitary Nearness Posets} \qquad \leftrightarrow \qquad \text{Compact } T_1 \text{ Spaces.}$$

Extensions

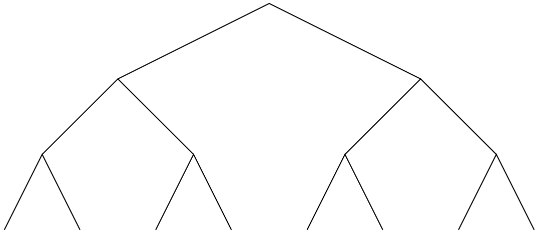
- It is then natural to investigate potential Wallman-type dualities for non-compact nearness spaces, e.g.

Star-Finitary Nearness Posets $\leftrightarrow$ Locally Compact T_1 Spaces.

- Using the Arhangelskii-Stone metrisation theorem, we also have have an analog for completely metrisable spaces, via regular Θ with a countable filter base (w.r.t. refinement).
- For the details see ArXiv:1902.07948 ‘Nearness Posets’.
- Aside: compact metric spaces are supercompact, i.e. they have a subbasis s.t. every cover has a 2-element subcover.
- Thus these correspond to 2-ary nearness posets.
- So all compact metric spaces arise as the spectrum of a countable graph, which could be worth exploring further.

Graded Posets

- Graded/ranked posets have a natural nearness structure coming from the rank levels, i.e. taking these as a base for Θ.
- Many natural examples of arise in this way.
- E.g. the standard basis of the Cantor space $\{0, 1\}^{\mathbb{N}}$ coming from finite initial sequences yields the complete binary tree:

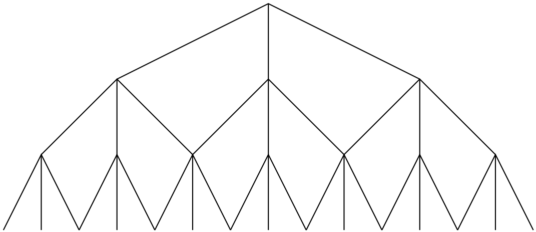


The Arc

- Similarly, the arc/interval $[0, 1]$, with the dyadic basis

$$\{(\tfrac{k-1}{2^n}, \tfrac{k+1}{2^n}) : k, n \in \mathbb{N} \text{ and } 1 < k < 2^n - 1\}$$

and $\{[0, \frac{1}{2^n}) : n \in \mathbb{N}\} \cup \{(1 - \frac{1}{2^n}, 1] : n \in \mathbb{N}\} \cup \{[0, 1]\}$ yields



Graded Posets $\leftrightarrow$ Compact T_1 Spaces

Theorem

Every second countable compact T_1 space is the spectrum of a countable graded poset with finite levels.

- Analogous to the fact compact Hausdorff spaces are all inverse limits of simplicial complexes (Freudanthal 1937).
- But this is not merely of theoretical interest - it suggests we could actually construct interesting spaces by first constructing an appropriate graded poset, e.g. by recursively defining the levels and the relations between them.
- E.g. for the pseudoarc, we could consider the category of finite paths/linear graphs, where the morphisms are relations between them that preserve and reflect the graph structure.
- This category has the amalgamation property and hence a Fraïssé sequence, which we combine to form a graded poset.
- The spectrum of this poset is precisely the pseudoarc.
- To obtain the Lelek fan, replace paths with rooted trees.