The Bohr compactification of an abelian group as a quotient of its Stone-Čech compactification

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The Stone-Čech compactification of a semigroup 1

For any (discrete) semigroup S, its Stone-Čech compactification $\boldsymbol{\beta} S$ admits a semigroup operation extending the original multiplication on S and turning it into the universal compact right topological semigroup densely extending S.

$$\boldsymbol{\beta} S = \{u \mid u \text{ is an ultrafilter on the set } S\}$$

For $u, v \in \boldsymbol{\beta} S$, $A \subseteq S$

$$A \in uv \iff \left\{s \in S \mid s^{-1}A \in v\right\} \in u$$

where, for $s \in S$,

$$s^{-1}A = L_s^{-1}[A] = \{x \in S \mid sx \in A\}$$

The sets $\{u \in \boldsymbol{\beta} S \mid A \in u\}$, with $A \subseteq S$, form a base of a compact hausdorff topology on S, consisting of clopen sets.

Identifying each $s \in S$ with the principal ultrafilter $\{A \subseteq S \mid s \in A\}$, S is embedded into $\boldsymbol{\beta} S$.

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$(\boldsymbol{\beta} S, \cdot)$ is a **right topological semigroup**, i.e., all the right shifts $R_v\colon \boldsymbol{\beta} S \to \boldsymbol{\beta} S$, $R_v(u) = uv$ are continuous.

$\boldsymbol{\beta} S$ has the following **universal property**:

Every homomorphism $h\colon S \to K$ from S to a compact hausdorff right topological semigroup K extends to a unique continuous homomorphism $\widetilde{h}\colon \boldsymbol{\beta} S \to K$; $\widetilde{h}$ is onto iff $h[S]$ is dense in K.

The semigroups $\boldsymbol{\beta} S$ have proved their usefulness, versatility and importance in various branches of mathematics.

In particular, the algebraic and topological structure of the semigroups $\boldsymbol{\beta}\mathbb{N}$ and $\boldsymbol{\beta}\mathbb{Z}$ has been spectacularly applied in proving a handful of striking Ramsey type combinatorial results in number theory.

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Hindman's Theorem
Let the set of all natural numbers be colored with finitely many colors. Then there is an infinite set $X \subseteq \mathbb{N}$ such that all finite sums of elements of X have the same color.

Crucial moment in the proof:

Existence of idempotent ultrafilters in $(\boldsymbol{\beta}\mathbb{N}, +)$

First Ellis' Theorem
Every compact right topological semigroup T contains an idempotent (i.e. an $e \in T$, such that $ee = e$).

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Theorem
Let T be a compact right topological semigroup and $\Theta = \Theta(T)$ denote the least closed congruence relation on T containing all the pairs (eu, u) where $u \in T$ is an arbitrary element and $e \in T$ is an idempotent. Then the quotient T/Θ is a compact right topological **group**. Moreover, if E is any closed congruence relation on T, then T/E is a right topological group iff $\Theta \subseteq E$.

Sketch of proof: For $v \in T$,
$Sv = R_v[T] = \{tv \mid t \in T\}$ is a compact subsemigroup of T.
By the first Ellis' theorem, it contains an idempotent
of the form $e = uv$, where $u \in T$.
Then $[u]_\Theta\,[v]_\Theta = [e]_\Theta$ is the unit in T/Θ, and
$[u]_\Theta$ is the left inverse of $[v]_\Theta$ in T/Θ.
Thus T/Θ is indeed a group.

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In particular, the quotient $\boldsymbol{\beta} S/\Theta(\boldsymbol{\beta} S)$ has the following **universal property**:

Let S be a (discrete) semigroup. Then every homomorphism $h\colon S \to K$ from S to a compact hausdorff right topological group K extends to a unique continuous homomorphism $h'\colon \boldsymbol{\beta} S/\Theta(\boldsymbol{\beta} S) \to K$; h' is onto iff $h[S]$ is dense in K.

The Bohr compactification of an abelian group 1

The Bohr compactification $\mathfrak{b}G$ of a locally compact abelian group G is the universal compact abelian group densely extending G.

It is of crucial importance in harmonic analysis, mainly as the tool enabling to treat the almost periodic functions on G through their (continuous) extensions to $\mathfrak{b}G$.

A bounded continuous function $f\colon G \to \mathbb{C}$ is **almost periodic** if the set $\{f_a \mid a \in G\}$ of its left shifts $f_a(x) = f(ax)$ is relatively compact in the banach space $C(G)$ with the norm $\|f\|_\infty = \sup_{x\in G} |f(x)|$.

Equivalently, $f \in C(G)$ is almost periodic iff it has a continuous extension to a function $f^\sharp\colon \mathfrak{b}G \to \mathbb{C}$.

The Bohr compactification of an abelian group 2

For an abelian group G, its dual $\widehat{G} = \mathrm{Hom}(G, \mathbb{T})$, where $\mathbb{T} \subseteq \mathbb{C}$ is the unit circle, is again an abelian group under the pointwise multiplication of characters $\gamma, \chi \in \widehat{G}$ given by

$$(\gamma\chi)(x) = \gamma(x)\chi(x) \quad (x \in G)$$

Being a closed subgroup of $\mathbb{T}^G$, $\widehat{G}$ is a compact hausdorff topological group.

Let $\widehat{G}_{\mathrm{d}}$ denote the dual of G endowed with the discrete topology.

Then the Bohr compactification of G can be defined as the dual $\mathfrak{b}G = \widehat{\widehat{G}_{\mathrm{d}}}$ of $\widehat{G}_{\mathrm{d}}$.

The Bohr compactification of an abelian group 3

$\mathfrak{b}G$ is a compact topological group and G can be canonically embedded into $\mathfrak{b}G$ as a dense subset, identifying any $x \in G$ with the character $x\colon \widehat{G}_{\mathrm{d}} \to \mathbb{T}$ of $\widehat{G}_{\mathrm{d}}$, given by $x(\gamma) = \gamma(x)$ for $\gamma \in \widehat{G}_{\mathrm{d}}$.

$\mathfrak{b}G$ has the following **universal property**:

Every homomorphism $h\colon G \to K$ from G to a compact hausdorff topological group K extends to a unique continuous homomorphism $h^\sharp\colon \mathfrak{b}G \to K$; $h^\sharp$ is onto iff $h[G]$ is dense in K.

$\mathfrak{b}G$ as quotient of $\boldsymbol{\beta}G$ 1

Since the Stone-Čech compactification is "more universal" than the Bohr one, there is a canonical continuous map $\xi\colon \boldsymbol{\beta}G \to \mathfrak{b}G$ such that $\xi(x) = x$ for $x \in G$. ξ is a surjective homomorphism.

$$\mathrm{Eq}(\xi) = \{(u, v) \in \boldsymbol{\beta}G \times \boldsymbol{\beta}G \mid \xi(u) = \xi(v)\}$$

is a closed congruence relation on $\boldsymbol{\beta}G$.

An ultrafilter $u \in \boldsymbol{\beta}G$ is called a **Schur ultrafilter** if

$$\forall A \in u \; \exists\, a, b \in A\colon ab \in A$$

Every idempotent ultrafilter is Schur.

Schur ultrafilters enable to generalize Ramsey type results like

Schur's Theorem
Let the set of all integers be colored with finitely many colors. Then there are $a, b \in \mathbb{Z}$ such that a, b and $a + b$ have all the same color.

$\mathfrak{b}G$ *as quotient of* $\boldsymbol{\beta}G$ *2*

We denote $A^{-1} = \{a^{-1} \mid a \in A\}$, for $A \subseteq G$,
and $u^{-1} = \{A^{-1} \mid A \in u\}$, for $u \in \boldsymbol{\beta}G$.

Protasov's Lemma
For any ultrafilter $u \in \boldsymbol{\beta}G$, uu^{-1} is Schur.

Theorem
Let G be a (discrete) group and $\varXi = \varXi(G)$ denote the least closed congruence relation on $\boldsymbol{\beta}G$ containing all the pairs $(u, 1)$ where $u \in \boldsymbol{\beta}G$ is a Schur ultrafilter. Then the quotient $\boldsymbol{\beta}G/\varXi$ is a compact **topological group**. Moreover, if E is any closed congruence relation on $\boldsymbol{\beta}G$, then $\boldsymbol{\beta}G/E$ is a topological group iff $\varXi \subseteq E$.

Corollary
Let G be a (discrete) abelian group. Then

$$\mathrm{Eq}(\xi) = \varXi(G) \quad \text{and} \quad \mathfrak{b}G \cong \boldsymbol{\beta}G/\varXi(G)$$

$\mathfrak{b}G$ *as quotient of* $\boldsymbol{\beta}G$ *3*

Sketch of proof: Obviously, $G/\varXi$ is a right topological group with continuous inverse map.
Hence it is a left topological group, as well.
The rest follows from the second Ellis' theorem.

Second Ellis' Theorem
Let G be both a right and left topological group and a hausdorff locally compact space. Then G is a topological group.

Thank you for your attention