Divisibility and diagonals in many-valued logic

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1. Divisibility

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Divisibility (1)

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$$a \wedge b \quad \leq \quad a \wedge (a \Rightarrow b) \quad \leq \quad a \wedge b$$

(from $a \wedge b \leq b$ get $b \leq (a \Rightarrow b)$)

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... in the powerset of a monoid with pointwise multiplication,

... in the monoid of relations on a set with usual relational composition,

... in the monoid of sup-morphisms on a complete lattice with usual composition.

Divisibility (2)

A residuated monoid $(M, \cdot, 1, \searrow, \swarrow)$ is **divisible** if

$$a \cdot (a \searrow b) = a \wedge b = (b \swarrow a) \cdot a$$

holds for all $a, b \in M$.

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A quantaloid $\mathcal{Q}$ is a category with hom-sup-lattices $\mathcal{Q}(X, Y)$ such that all $- \circ f$ and $g \circ -$ preserve suprema; it is therefore also residuated:

$$g \circ f \leq h \iff f \leq (g \searrow h) \iff g \leq (h \swarrow f)$$

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It now makes perfect sense to say that a quantaloid $\mathcal{Q}$ is **divisible** if

$$g \circ (g \searrow f) = f \wedge g = (f \swarrow g) \circ g$$

for every pair $f, g \colon X \to Y$ of parallel arrows in $\mathcal{Q}$.

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- Any locale is divisible.
- A left-continuous t-norm $([0,1], \star, 1)$ is (by definition) a commutative, integral, ordered monoid with left-continuous multiplication; this is precisely an integral, commutative quantale on $([0,1], \bigvee)$. Such a left-continuous t-norm is (also right-)continuous if and only if (as a quantale) it is divisible.

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- Lawvere's quantale of real numbers $([0,\infty],\bigwedge,+,0)$ is divisible; it is isomorphic to the (obviously continuous) product t-norm $([0,1],\bigvee,\cdot,1)$.
- Any non-(right-)continuous left-continuous t-norm thus provides an example of an integral and localic quantale which is not divisible (e.g. the "nilpotent minimum t-norm").

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As a well-known case in point, in any category $\mathcal{C}$, if $f^2 = f$ and $g^2 = g$ are two **idempotents**, then we say that m is a **map** from f to g if

$$\begin{array}{ccc} A & \xrightarrow{m} & B \\ {\scriptstyle f}\downarrow & \searrow^{m} & \downarrow{\scriptstyle g} \\ A & \xrightarrow[m]{} & B \end{array}$$

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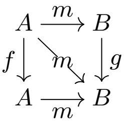
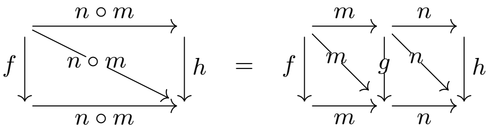


A new category $\mathcal{I}(\mathcal{C})$ of maps between idempotents in $\mathcal{C}$ is defined by the obvious composition rule



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$$\begin{array}{ccc} & \xrightarrow{n \circ m} & \\ {\scriptstyle f}\downarrow & \searrow^{n \circ m} & \downarrow{\scriptstyle h} \\ & \xrightarrow[n \circ m]{} & \end{array} \quad = \quad \begin{array}{ccccc} & \xrightarrow{m} & & \xrightarrow{n} & \\ {\scriptstyle f}\downarrow & \searrow^{m} & \downarrow{\scriptstyle g} & \searrow^{n} & \downarrow{\scriptstyle h} \\ & \xrightarrow[m]{} & & \xrightarrow[n]{} & \end{array}$$

with identities $\begin{array}{ccc} & \xrightarrow{f} & \\ {\scriptstyle f}\downarrow & \searrow^{f} & \downarrow{\scriptstyle f} \\ & \xrightarrow[f]{} & \end{array}$ for each $f^2 = f$.

Diagonals (2)

There is a full embedding

$$I\colon \mathcal{C} \to \mathcal{I}(\mathcal{C})\colon \left(A \xrightarrow{m} B \right) \mapsto \left(\begin{array}{ccc} A & \xrightarrow{m} & B \\ {\scriptstyle 1_A}\downarrow & \overset{m}{\searrow} & \downarrow{\scriptstyle 1_B} \\ A & \xrightarrow[m]{} & B \end{array} \right)$$

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displaying $\mathcal{I}(\mathcal{C})$ to be the universal "split-idempotent" completion of $\mathcal{C}$:

$$\text{if } f^2 = f \text{ in } \mathcal{C} \text{ then } \quad \begin{array}{ccc} A & \xrightarrow{f} & A \\ {\scriptstyle 1_A}\downarrow & \searrow^{f} & \downarrow{\scriptstyle 1_A} \\ A & \xrightarrow[f]{} & A \end{array} \quad = \quad \begin{array}{ccccc} A & \xrightarrow{f} & A & \xrightarrow{f} & A \\ {\scriptstyle 1_A}\downarrow & \searrow^{f} & \downarrow{\scriptstyle f} & \searrow^{f} & \downarrow{\scriptstyle 1_A} \\ A & \xrightarrow[f]{} & A & \xrightarrow[f]{} & A \end{array} \quad \text{in } \mathcal{I}(\mathcal{C})$$

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(Actually, all idempotents split in $\mathcal{I}(\mathcal{C})$.)

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... but for our purposes, it is not yet big enough.

Diagonals (3)

In any category $\mathcal{C}$, say that d is a **diagonal** from f to g

$$\begin{array}{ccc} A_0 & & B_0 \\ {\scriptstyle f}\downarrow & \overset{d}{\searrow} & \downarrow{\scriptstyle g} \\ A_1 & & B_1 \end{array}$$

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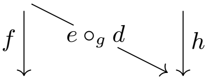
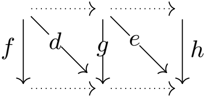
A new category $\mathcal{D}(\mathcal{C})$ of diagonals in $\mathcal{C}$ is defined by the composition rule



$=$ any path from UL to LR in



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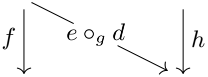

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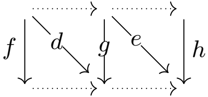

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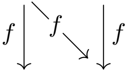

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$$\text{for any } f \text{ in } \mathcal{C}, \quad \begin{array}{ccc} A & & B \\ {\scriptstyle 1_A}\downarrow & \searrow^{f} & \downarrow{\scriptstyle 1_B} \\ A & & B \end{array} \;=\; \begin{array}{ccccc} A & & A & & B \\ {\scriptstyle 1_A}\downarrow & \searrow^{f} & \downarrow{\scriptstyle f} & \searrow^{f} & \downarrow{\scriptstyle 1_B} \\ A & & B & & B \end{array} \quad \text{in } \mathcal{D}(\mathcal{C})$$

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The splitting of idempotents in $\mathcal{C}$ is a full subcategory of $\mathcal{D}(\mathcal{C})$:

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(Actually all arrows in $\mathcal{D}(\mathcal{C})$ have an image factorisation; this leads to a monadic characterisation of proper factorisation systems.)

The splitting of idempotents in $\mathcal{C}$ is a full subcategory of $\mathcal{D}(\mathcal{C})$:

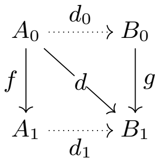
$$\mathcal{C} \longrightarrow \mathcal{I}(\mathcal{C}) \longrightarrow \mathcal{D}(\mathcal{C})$$

Note: even for a monoid M, both $\mathcal{I}(M)$ and $\mathcal{D}(M)$ are (many-object) categories.

Diagonals (5)

In any quantaloid $\mathcal{Q}$, making use of residuation,

$\exists d_0, d_1 :$



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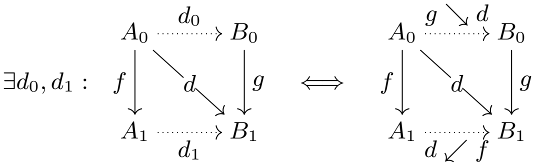

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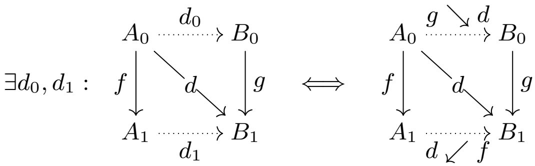

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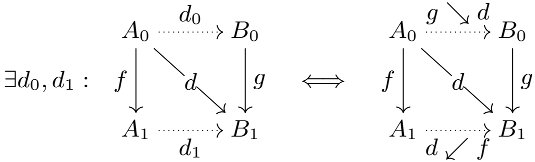

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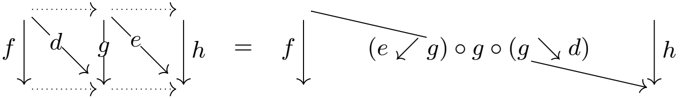

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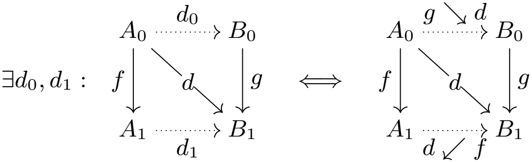

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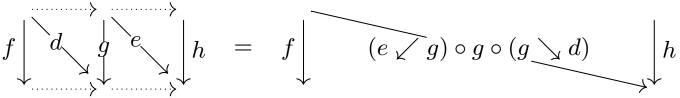

This holds *a fortiori* for $\mathcal{I}(\mathcal{Q})$ too, and the full embeddings are indeed quantaloid homomorphisms:

$$\mathcal{Q} \longrightarrow \mathcal{I}(\mathcal{Q}) \longrightarrow \mathcal{D}(\mathcal{Q})$$

Diagonals (6)

Recall, a quantaloid $\mathcal{Q}$ is **divisible** if, for every $f, g\colon X \to Y$,

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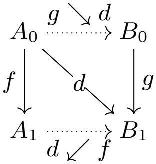
And d is a **diagonal** from f to g precisely when  .

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Proof:

$\boxed{\Rightarrow}$ If $\mathcal{Q}$ is divisible then it is integral; so when $g \circ x = d = y \circ f$ then surely $d \leq f \wedge g$; and conversely, from $d \leq f \wedge g \leq f$ we get $d = d \wedge f = (d \swarrow f) \circ f$ and similarly $d = g \circ (g \searrow d)$.

$\boxed{\Leftarrow}$ If $\mathcal{D}(\mathcal{Q})(f,g) = {\downarrow} f \wedge g$ then $\mathcal{Q}$ is integral because $\mathcal{Q}(X,X) = \mathcal{D}(\mathcal{Q})(1_X, 1_X) = {\downarrow} 1_X$; but also $g \circ x = f \wedge g = y \circ f$, which implies $x \leq g \searrow f$ and $y \leq g \swarrow f$ and from that also $f \wedge g \leq g(g \searrow f)$ and $f \wedge g \leq (f \swarrow g)g$; the other inequation holds by integrality, so $g(g \searrow f) = f \wedge g = (f \swarrow g)g$.

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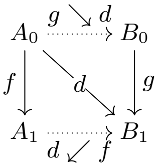

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Computations with $\mathcal{D}(\mathcal{Q})$ thus simplify a great deal whenever $\mathcal{Q}$ is a divisible quantaloid: because the hom-sup-lattices are easy, because the composition law is easy!

This applies to any divisible quantale Q—which is of use in many-valued logic.

3. Many-valued logic

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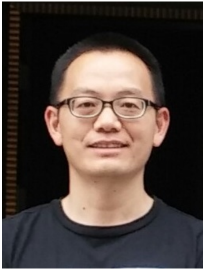


Many-valued logic (1)

An order $\mathbb{X} = (X, \leq)$ is a set together with a binary predicate

$$\mathbb{X}\colon X \times X \to \{\bot, \top\}\colon (x, y) \mapsto \begin{cases} \top \text{ if } x \leq y \\ \bot \text{ if } x \not\leq y \end{cases}$$

such that

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There is a very rich theory of Q-enriched categories, functors and distributors, which thus – at first sight – caters for a theory of "many-valued orders".

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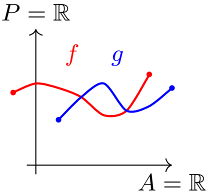
However...

Many-valued logic (2)

Let A be a set and $(P, \leq)$ an order, and consider the set

$$X = \{f \colon S \to P \text{ is a function} \mid S \subseteq A\}$$

of partial functions from A to P.



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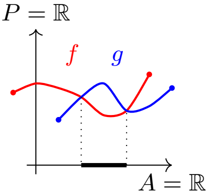

To compare partial functions f and g, it is most natural to compute the "extent to which f is smaller than g":

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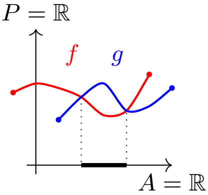

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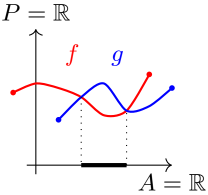

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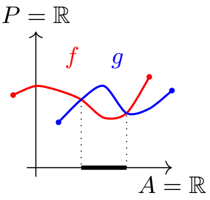

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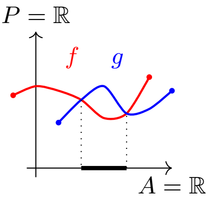

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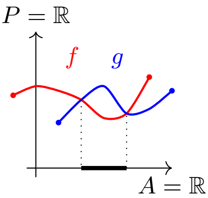

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Quantaloids, diagonals and divisibility to the rescue!

Many-valued logic (3)

Let $\mathcal{Q}$ be a (small) quantaloid. A $\mathcal{Q}$-**enriched category** $\mathbb{C}$ consists of:

- a set $\mathbb{C}_0$,
- a unary predicate $t\colon \mathbb{C}_0 \to \mathsf{obj}(\mathcal{Q})$,
- a binary predicate $\mathbb{C}\colon \mathbb{C}_0 \times \mathbb{C}_0 \to \mathsf{arr}(\mathcal{Q})$

for which we have:

- $\mathbb{C}(y, x)\colon tx \to ty$,
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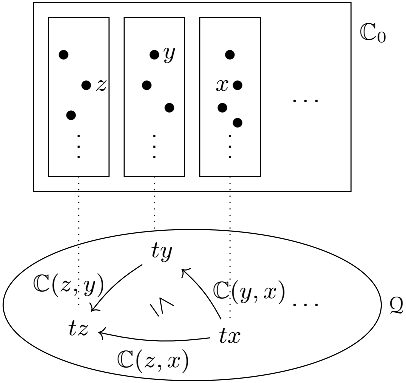

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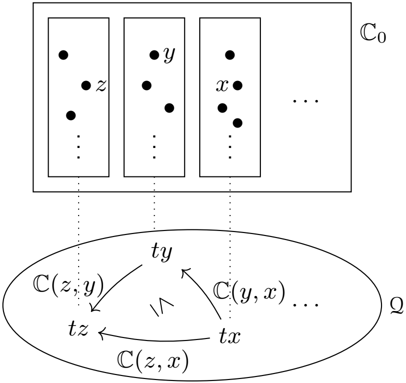

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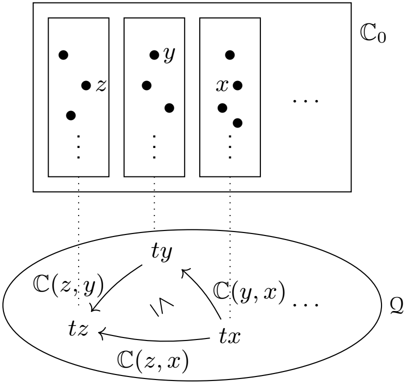

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But how can this help us with the previous (and other) examples?

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Let Q be a divisible, commutative quantale, and $\mathcal{D}(Q)$ the quantaloid of diagonals.

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Similar simplifications can be done for the notion of $\mathcal{D}(Q)$-enriched functor and distributor—for indeed, we have the complete quantaloid-enriched yoga at our disposal.

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Recall, for A a set and $(P, \leq)$ is order, we wish to consider the set

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With a bit more quantaloid-enriched category theory, one can deal with sheaves on a locale in this way.

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Quantaloid-enriched category theory can be put to use here, in particular to deal with Cauchy completion, exponentiability, Hausdorff distance, and more.

4. Summary

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- Such a "partial Q-category" is a many-valued order relation on a set of partially defined elements.
- Applied to a continuous t-norm, this seems to provide a useful notion of "fuzzy (pre)order" (on "fuzzy elements"!), but only further research and more examples can tell.