The logic of categories and informational entropy

Willem conradie
joint work with
A. Craig, A. Palmigiano and N. M. Wijnberg.

18 June 2019

TACL, Nice

1

Lattice-based modal logic

The language $\mathcal{L}$ of the basic normal non-distributive modal logic:

$$\varphi := \bot \mid \top \mid p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Box\varphi \mid \Diamond\varphi,$$

where $p \in \mathsf{Prop}$. The *basic*, or minimal normal $\mathcal{L}$-logic is a set $\mathbf{L}$ of sequents $\phi \vdash \psi$ with $\phi, \psi \in \mathcal{L}$, containing the following axioms:

$$
\begin{array}{llll}
p \vdash p, & \bot \vdash p, & p \vdash \top, & \\
p \vdash p \vee q, & q \vdash p \vee q, & p \wedge q \vdash p, & p \wedge q \vdash q, \\
\top \vdash \Box\top, & \Box p \wedge \Box q \vdash \Box(p \wedge q), & \Diamond\bot \vdash \bot, & \Diamond p \vee \Diamond q \vdash \Diamond(p \vee q)
\end{array}
$$

and closed under the following inference rules:

$$
\frac{\phi \vdash \chi \quad \chi \vdash \psi}{\phi \vdash \psi} \qquad
\frac{\phi \vdash \psi}{\phi\,(\chi/p) \vdash \psi\,(\chi/p)} \qquad
\frac{\chi \vdash \phi \quad \chi \vdash \psi}{\chi \vdash \phi \wedge \psi} \qquad
\frac{\phi \vdash \chi \quad \psi \vdash \chi}{\phi \vee \psi \vdash \chi}
$$

$$
\frac{\phi \vdash \psi}{\Box\phi \vdash \Box\psi} \qquad
\frac{\phi \vdash \psi}{\Diamond\phi \vdash \Diamond\psi}
$$

Introduction

Problem: Understanding relational semantics for lattice-based (e.g. substructural) logics:
Two options:

- Polarity-based (two-sorted).
- Graph based (single sorted).

Notation: Let $T \subseteq U \times V$, and any $U' \subseteq U$ and $V' \subseteq V$.

$$\begin{aligned} T^{(1)}[U'] &= \{v \mid \forall u(u \in U' \Rightarrow uTv)\} \\ T^{(0)}[V'] &= \{u \mid \forall v(v \in V' \Rightarrow uTv)\}. \end{aligned}$$

$$\begin{aligned} T^{[1]}[U'] &= \{v \mid \forall u(u \in U' \Rightarrow uT^c v)\} \\ T^{[0]}[V'] &= \{u \mid \forall v(v \in V' \Rightarrow uT^c v)\}. \end{aligned}$$

Two-sorted semantics for lattice-based modal logics

Polarity. $\mathbb{P} = (A, X, \mathrm{I})$ with A and X sets and $\mathrm{I} \subseteq A \times X$.

Galois connection. $(\cdot)^{(1)} : \mathcal{P}A \to \mathcal{P}X$ and $(\cdot)^{(0)} : \mathcal{P}X \to \mathcal{P}A$ s.t. for all $B \subseteq A$ and $Y \subseteq X$,

- $B^{(1)} := \{x \in X \mid \forall a(a \in B \to a\mathrm{I}x)\}$,
- $Y^{(0)} := \{a \in A \mid \forall x(x \in Y \to a\mathrm{I}x)\}$.

Closed sets. $B = B^{(10)}$ and $Y = Y^{(01)}$.

Lattice of closed sets. Let $C(A)$ (resp. $C(X)$) be the closed subsets of A (resp. X).

$$\mathbb{P}^{+} = (C(A), \bigcap, \bigvee, \varnothing^{(10)}, A) \cong^{\partial} (C(X), \bigcap, \bigvee, \varnothing^{(01)}, X).$$

Concept lattice of $\mathbb{P}$. Lattice of tuples $C = ([\![C]\!], (\![C]\!))$ s.t.

$$[\![C]\!] = (\![C]\!)^{(0)} \quad \text{and} \quad (\![C]\!) = [\![C]\!]^{(1)}.$$

Polarity-based frames and models

Polarity-based frame. $\mathbb{F} = (\mathbb{P}, R)$ such that

- $\mathbb{P} = (A, X, \mathrm{I})$ is a polarity
- $R \subseteq A \times X$
- $R^{(1)}[b]$ and $R^{(0)}[y]$ are closed sets, for all $b \in A$ and $y \in X$.

Polarity-based models. $\mathbb{M} = (\mathbb{F}, V)$ s.t.

- $\mathbb{F}$ a polarity-based frame
- for all $p \in \mathbf{AtProp}$,

$$V(p) = ([\![p]\!], (\![p]\!))$$

with $[\![p]\!] = (\![p]\!)^{(0)}$ and $(\![p]\!) = [\![p]\!]^{(1)}$.

Interpretation of lattice-based modal logic on RS-frames

$\mathbb{M}, a \Vdash \bot$	never	$\mathbb{M}, x \succ \bot$	always
$\mathbb{M}, a \Vdash \top$	always	$\mathbb{M}, x \succ \top$	never

$\mathbb{M}, a \Vdash p$ iff $a \in [\![p]\!]$ $\quad$ $\mathbb{M}, x \succ p$ iff $x \in (\![p]\!)$

$\mathbb{M}, a \Vdash \phi \wedge \psi$ iff $\mathbb{M}, a \Vdash \phi$ and $\mathbb{M}, a \Vdash \psi$
$\mathbb{M}, x \succ \phi \wedge \psi$ iff for all $a \in A$, if $\mathbb{M}, a \Vdash \phi \wedge \psi$, then $a\mathrm{I}x$

$\mathbb{M}, a \Vdash \phi \vee \psi$ iff for all $x \in X$, if $\mathbb{M}, x \succ \phi \vee \psi$, then $a\mathrm{I}x$
$\mathbb{M}, x \succ \phi \vee \psi$ iff $\mathbb{M}, x \succ \phi$ and $\mathbb{M}, x \succ \psi$

$\mathbb{M}, a \Vdash \Box\phi$ iff for all $x \in X$, if $\mathbb{M}, x \succ \phi$, then aRx
$\mathbb{M}, x \succ \Box\phi$ iff for all $a \in A$, if $\mathbb{M}, a \Vdash \Box\phi$, then $a\mathrm{I}x$

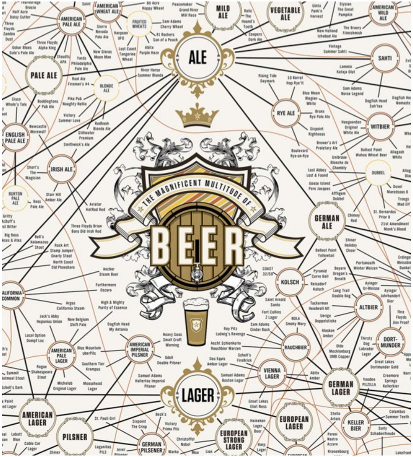
Categorization theory

From Wikipedia:

Categorization is the process in which ideas and objects are recognized, differentiated, and understood.

Ideally, a category illuminates a relationship between the subjects and objects of knowledge.

Categorization is fundamental in language, prediction, inference, decision making and in all kinds of environmental interaction.



Categorization theory and RS-models via Formal Concept Analysis

Let $\mathbb{F} = (\mathbb{P}, R)$ with

- $\mathbb{P} = (A, X, \mathrm{I})$ database
- A set of objects (e.g. car models currently on sale)
- X set of features (e.g. electric, 3 doors, red...)
- I incidence relation: $a\mathrm{I}x$ iff object a has feature x
- $R \subseteq A \times X$ knowledge/perception/beliefs of a given agent:

  aRx iff object a has feature x according to the agent
- a^1 set of features of object a
- x^0 set of objects having feature x
- B^1 set of features shared by all objects in B
- Y^0 set of objects satisfying all features in Y
- $\mathbb{P}^+$ concept lattice arising from database $\mathbb{P}$

Categories as social constructs

Social interaction is key to categorization theory:

- categories arise from factual information about the world.
- **However**, what they mean critically depends on how people **perceive** them and **agree** about them

Three aspects of categorization theory:

- factual truth
- subjective perception / knowledge / beliefs
- social interaction

Epistemic interpretation of $\Box$

In an RS-frame $\mathbb{F} = (\mathbb{P}, R)$:

- $R \subseteq A \times X$ encodes perception of a given agent about objects and their features
- aRx reads 'object a has feature x according to the agent'
- $\Box\phi$ reads 'category which the agent understands as ϕ'

Example: Factivity of knowledge. $\Box\phi \leq \phi$

$$
\begin{array}{ll}
 & \forall p(\Box p \leq p) \\
\text{iff} & \forall \mathbf{m}(\Box \mathbf{m} \leq \mathbf{m}) \\
\text{iff} & \forall a \forall \mathbf{m}[\mathrm{ST}_a(\Box \mathbf{m}) \to \mathrm{ST}_a(\mathbf{m})] \\
\text{iff} & \forall a \forall m(aRm \to a\mathrm{I}m),
\end{array}
$$

if a has m according to the agent, then a has m in reality

Graphs and lattices

A reflexive graph is a structure $\mathbb{X} = (Z, E)$.
Any graph $\mathbb{X} = (Z, E)$ defines the polarity $\mathbb{P}_{\mathbb{X}} = (Z, Z, E^c)$.
The complete lattice $\mathbb{X}^+$ associated with a graph $\mathbb{X}$ is defined as the concept lattice of $\mathbb{P}_{\mathbb{X}}$.
$\mathbb{L}$ a lattice. $\mathsf{Flt}(\mathbb{L})$: filters $\mathbb{L}$. $\mathsf{Idl}(\mathbb{L})$: filters $\mathbb{L}$.
The graph associated with $\mathbb{L}$ is $\mathbb{X}_{\mathbb{L}} := (Z, E)$ where $Z := \{(F, J) \in \mathsf{Flt}(\mathbb{L}) \times \mathsf{Idl}(\mathbb{L}) \mid F \cap J = \varnothing\}$.
For $z \in Z$, we denote by F_z the filter part of z and by J_z the ideal part of z.
The (reflexive) E relation is defined by zEz' if and only if $F_z \cap J_{z'} = \emptyset$.

Proposition [Craig & Havier, 2014]

For any lattice $\mathbb{L}$, the complete lattice ${\mathbb{X}_{\mathbb{L}}}^+$ is the canonical extension of $\mathbb{L}$.

Graph-based frames

Definition

A *graph-based* $\mathcal{L}$-**frame** is a structure $\mathbb{F} = (\mathbb{X}, R_\Diamond, R_\Box)$ where

- $\mathbb{X} = (Z, E)$ is a reflexive graph
- $R_\Diamond$ and $R_\Box$ are binary relations on Z satisfying the following E-*compatibility* conditions:

$$(R_\Box^{[0]}[y])^{[10]} \subseteq R_\Box^{[0]}[y] \qquad (R_\Box^{[1]}[b])^{[01]} \subseteq R_\Box^{[1]}[b]$$
$$(R_\Diamond^{[0]}[b])^{[10]} \subseteq R_\Diamond^{[0]}[b] \qquad (R_\Diamond^{[1]}[y])^{[01]} \subseteq R_\Diamond^{[1]}[y].$$

Graph-based frames and $\mathcal{L}$-algebras

The complex algebra of a graph-based $\mathcal{L}$-frame $\mathbb{F} = (\mathbb{X}, R_\Diamond, R_\Box)$: the complete $\mathcal{L}$-algebra $\mathbb{F}^+ = (\mathbb{X}^+, [R_\Box], \langle R_\Diamond \rangle)$, where:

- $\mathbb{X}^+$ is the concept lattice of $\mathbb{P}_\mathbb{X}$
- for every $c = (\llbracket c \rrbracket, (\!| c |\!)) \in \mathbb{P}^+_\mathbb{X}$,

$$[R_\Box]c := (R_\Box^{[0]}[(\!| c |\!)], (R_\Box^{[0]}[(\!| c |\!)])^{[1]})$$

and

$$\langle R_\Diamond \rangle c := ((R_\Diamond^{[0]}[\llbracket c \rrbracket])^{[0]}, R_\Diamond^{[0]}[\llbracket c \rrbracket])$$

Lemma

The algebra $\mathbb{F}^+ = (\mathbb{X}^+, [R_\Box], \langle R_\Diamond \rangle)$ is a complete lattice expansion such that $[R_\Box]$ is completely meet-preserving and $\langle R_\Diamond \rangle$ is completely join-preserving.

Graph-based models

Definition

A graph-based $\mathcal{L}$-model is a tuple $\mathbb{M} = (\mathbb{F}, V)$ where $\mathbb{F}$ is a graph-based $\mathcal{L}$-frame and V : Prop $\to \mathbb{F}^+$.

Since $V(p)$ is a formal concept, we will write $V(p) = ([\![p]\!], (\![p]\!))$.

Extended V compositionally to all $\mathcal{L}$-formulas as follows:

$$\begin{array}{rcl}
V(p) & = & ([\![p]\!], (\![p]\!)) \\
V(\top) & = & (Z, \emptyset) \\
V(\bot) & = & (\emptyset, Z) \\
V(\phi \wedge \psi) & = & ([\![\phi]\!] \cap [\![\psi]\!], ([\![\phi]\!] \cap [\![\psi]\!])^{[1]}) \\
V(\phi \vee \psi) & = & (((\![\phi]\!) \cap (\![\psi]\!))^{[0]}, (\![\phi]\!) \cap (\![\psi]\!)) \\
V(\Box\phi) & = & (R_{\Box}^{[0]}[(\![\phi]\!)], (R_{\Box}^{[0]}[(\![\phi]\!)])^{[1]}) \\
V(\Diamond\phi) & = & ((R_{\Diamond}^{[0]}[[\![\phi]\!]])^{[0]}, R_{\Diamond}^{[0]}[[\![\phi]\!]])
\end{array}$$

Graph-based semantics

$\mathbb{M}, z \Vdash \bot$		never
$\mathbb{M}, z \succ \bot$		always
$\mathbb{M}, z \Vdash \top$		always
$\mathbb{M}, z \succ \top$		never
$\mathbb{M}, z \Vdash p$	iff	$z \in [\![p]\!]$
$\mathbb{M}, z \succ p$	iff	$\forall z'[z'Ez \Rightarrow z' \nVdash p]$
$\mathbb{M}, z \succ \phi \vee \psi$	iff	$\mathbb{M}, z \succ \phi$ and $\mathbb{M}, z \succ \psi$
$\mathbb{M}, z \Vdash \phi \vee \psi$	iff	$\forall z'[zEz' \Rightarrow \mathbb{M}, z' \not\succ \phi \vee \psi]$
$\mathbb{M}, z \Vdash \phi \wedge \psi$	iff	$\mathbb{M}, z \Vdash \phi$ and $\mathbb{M}, z \Vdash \psi$
$\mathbb{M}, z \succ \phi \wedge \psi$	iff	$\forall z'[z'Ez \Rightarrow \mathbb{M}, z' \nVdash \phi \wedge \psi]$
$\mathbb{M}, z \succ \Diamond\phi$	iff	$\forall z'[zR_{\Diamond}z' \Rightarrow \mathbb{M}, z' \nVdash \phi]$
$\mathbb{M}, z \Vdash \Diamond\phi$	iff	$\forall z'[zEz' \Rightarrow \mathbb{M}, z' \not\succ \Diamond\phi]$
$\mathbb{M}, z \Vdash \Box\psi$	iff	$\forall z'[zR_{\Box}z' \Rightarrow \mathbb{M}, z' \not\succ \psi]$
$\mathbb{M}, z \succ \Box\psi$	iff	$\forall z'[z'Ez \Rightarrow \mathbb{M}, z' \nVdash \Box\psi]$

Graph-based semantics (2)

An $\mathcal{L}$-sequent $\phi \vdash \psi$ is true in $\mathbb{M}$, denoted $\mathbb{M} \models \phi \vdash \psi$, if for all $z, z' \in Z$, if $\mathbb{M}, z \Vdash \phi$ and $\mathbb{M}, z' \succ \psi$ then $zE^c z'$.

An $\mathcal{L}$-sequent $\phi \vdash \psi$ is valid in $\mathbb{F}$, denoted $\mathbb{F} \models \phi \vdash \psi$, if it is true in every model based on $\mathbb{F}$.

Theorem

The basic non-distributive modal logic $\mathbf{L}$ *is sound and complete complete w.r.t. the class of graph-based* $\mathcal{L}$*-frames.*

Correspondence — E-composition

Definition

For any graph $\mathbb{X} = (Z, E)$ and relations $R, S \subseteq Z \times Z$, the E-compositions of R and S are the relations $R \circ_E S \subseteq Z \times Z$ and $R \bullet_E S \subseteq Z \times Z$ defined as follows: for any $a, x \in Z$,

$$x(R \circ_E S)a \quad \text{iff} \quad \exists b(xRb \;\&\; E^{(1)}[b] \subseteq S^{(0)}[a]).$$

$$a(R \bullet_E S)x \quad \text{iff} \quad \exists y(aRy \;\&\; E^{(0)}[y] \subseteq S^{(0)}[x]).$$

When $E = \Delta$, E-composition = ordinary relational composition.

Correspondence — E-parametric conditions

Proposition

For any graph-based $\mathcal{L}$-frame $\mathbb{F} = (\mathbb{X}, R_\Box, R_\Diamond)$,

1. $\mathbb{F} \models \Box\phi \vdash \phi$ iff $E \subseteq R_\Box$ ($R_\Box$ is E-reflexive).
2. $\mathbb{F} \models \phi \vdash \Diamond\phi$ iff $E \subseteq R_\blacksquare$ ($R_\Diamond$ is E-reflexive).
3. $\mathbb{F} \models \Box\phi \vdash \Box\Box\phi$ iff $R_\Box \bullet_E R_\Box \subseteq R_\Box$ ($R_\Box$ is $E_\bullet$-transitive).
4. $\mathbb{F} \models \Diamond\Diamond\phi \vdash \Diamond\phi$ iff $R_\Diamond \circ_E R_\Diamond \subseteq R_\Diamond$ ($R_\Diamond$ is $E_\circ$-transitive).
5. $\mathbb{F} \models \phi \vdash \Box\phi$ iff $R_\Box \subseteq E$ ($R_\Box$ is sub-E).
6. $\mathbb{F} \models \Diamond\phi \vdash \phi$ iff $R_\blacksquare \subseteq E$ ($R_\Diamond$ is sub-E)

Interpretation

$\mathbb{F} = (Z, E, R_\Diamond, R_\Box)$

- Z a set of states
- E and indiscernibility relation - inherent limits to knowability.
  1. $a^{[1]}$ — states not indeclinable from a
  2. $a^{[10]}$ — horizon to the possibility of completely 'knowing' a.
  3. horizon could be epistemic, cognitive, technological, or evidential.
  4. $E := \Delta$ represents limit case in which $a^{[10]} = \{a\}$.
- e.g. disjunction becomes weaker: $[\![\phi \vee \psi]\!] = ((\![\phi]\!) \cap (\![\psi]\!))^{[0]}$ requires a state z to satisfy $\phi \vee \psi$ exactly when z can be told apart from any state that refutes both ϕ and ψ.
- $R_\Diamond$ and $R_\Box$ subjective indiscernibility.