Exact and Fitted Sublocales

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Background

- Frames are dual point-free spaces
- Even classical spaces have more point-free sub-*spaces* than pointed subspaces — called sublocales
- The structure of the sublocales is interesting and useful to understand
- $S(L)$ — the lattice of sublocales — is a co-frame that is fairly well understood
- Today I talk about the strcuture of two classes of special sub-locales:
  - $S_c(L)$ — poset of joins of closed sublocales
  - $S_o(L)$ — poset of meets of open sublocales

Motivation

- Isbell introduced two weak separation properties (much weaker than regularity):
  - Subfitness: every open sublocale is a join of closed ones
  - Fitness: every closed sublocale is a meet of open ones
- Fitness is equivalent to *every* sublocale being a meet of opens
- Though it may not be clear, fitness really is properly stronger than subfitness
- To understand these, we need better tools to understand the structure of $\mathsf{S}_c(L)$ and $\mathsf{S}_o(L)$.

The main aims for today

- Define two posets of filters on L:
  - $\mathrm{Filt}_e(L)$: The poset of *exact* filters
  - $\mathrm{Filt}_f(L)$: The poset of *fitted* filters
- Show[1] that each is a sublocale of the frame of upsets of L: $\mathrm{Up}(L)$
- Show that $S_c(L) \simeq \mathrm{Filt}_e(L)$
- Show that $S_o(L) \simeq^{\partial} \mathrm{Filt}_f(L)$
- Prove some nice properties in the special cases of subfit and fit frames.

[1] When I say "show" in a 20 minute talk, I mean "assert".

Just what is a sublocale?

- A function $f\colon L \to M$ between frames (locales) is a locale map if
  - f preserves $\bigwedge$ — iff it has a lower adjoint given by

$$f_*(y) = \bigwedge\{x \in L \mid y \leq f(x)\}$$

  - $\top = f(a)$ implies $\top = a$ — iff f_* preserves $\top$
  - if $b_0 \wedge b_1 \leq f(a)$, then for some a_0, a_1:
    - $b_0 \leq f(a_0)$
    - $b_1 \leq f(a_1)$ and
    - $a_0 \wedge a_1 \leq a$

    — iff f_* preserves $\wedge$.
- A sublocale of L is a subset S for which inclusion is a locale map.
- Or S is closed under $\bigwedge$ and under "Heyting inflation" for any $a \in L$:

$$x \mapsto a \to x$$

Closed and open sublocales

Definitions

For $a \in L$, we define closed and open sublocales

- the associated closed sublocale is $c(a) = {\uparrow}a = \{x \vee a \mid x \in L\}$
- the associated open sublocale is $o(a) = \{a \to x \mid x \in L\}$

Basic observations

- The poset of closed sublocales is closed under finite joins and arbitrary meets
- $a \mapsto c(a)$ is order reversing sending finite meets to joins and arbitrary joins to meets
- The poset of open sublocales is closed under finite meets and arbitrary joins
- $a \mapsto o(a)$ preserves finite meets and arbitrary joins.

$\mathrm{S}_c(L)$ and $\mathrm{S}_o(L)$

Recall the definitions:

- $\mathrm{S}_c(L) =$ poset of joins of closed sublocales
- $\mathrm{S}_o(L) =$ poset of meets of open sublocales

Observations

- $\mathrm{S}_o(L) = \mathrm{S}(L)$ iff L is fit
- $\mathrm{S}_c(L) = \mathrm{S}(L)$ implies L is subfit, but this is stronger that subfit

Joins of closed sublocales

For a frame L, consider the maps $J\colon \mathrm{Up}(L) \to S(L)$, $U\colon S(L) \to \mathrm{Up}(L)$

$$J(A) = \bigsqcup_{a \in A} c(a)$$
$$U(S) = \{a \in L \mid c(a) \subseteq S\}$$

$$\mathrm{Up}(L) \underset{U}{\overset{J}{\rightleftarrows}} \mathrm{S}(L) \quad (J \dashv U)$$

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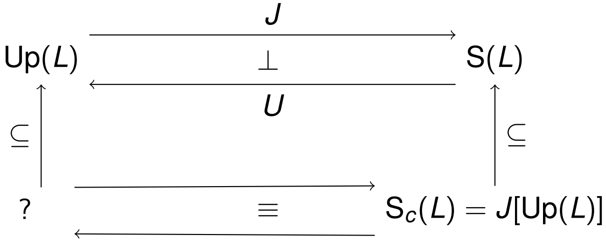

Which up sets correspond to $\mathrm{S}_c(L)$?

Frames that are sublocales of $\mathrm{Up}(A)$

In a join semilattice A:

- $\mathrm{Up}(A)$ is a completely distributive lattice, hence is a frame.
- $U \to V = \{a \in A \mid \forall b \in U, a \vee b \in V\}$.

— generalizes to posets, but we have no need here.

So $\mathcal{X} \subseteq \mathrm{Up}(A)$ is a sublocale of $\mathrm{Up}(A)$ iff

- $\mathcal{X}$ is closed under intersections
- For any $U \in \mathrm{Up}(A)$ and $V \in \mathcal{X}$, $U \to V \in \mathcal{X}$.

Example

On a distributive lattice $\mathrm{Filt}(A)$ is a sublocale of $\mathrm{Up}(A)$. One just checks that if U is an upset and F is a filter, then $U \to F$ is a filter.

Exact meets and joins

Definition

In a join semilattice A, a subset $B \subseteq A$ has an exact meet iff

- $\bigwedge B$ exists
- for all $a \in A$, $a \vee \bigwedge B = \bigwedge_{b \in B}(a \vee b)$.

In a meet semilattice, *exact joins* are defined dually.

Examples

- A distributive lattice is a join (meet) semilattice where all finite subsets have exact meets (joins).
- A frame is a meet semilattice where all subsets have exact joins.
- For any frame L, all meets are exact in $S(L)$. Hence $S(L)$ is a co-frame.

Exact filters

Exact meets lead us to exact filters:

- $F \subseteq A$ so that $B \subseteq F$ has an exact meet, then $\bigwedge B \in F$
- Let $\mathrm{Filt}_e(A)$ be the exact filters ordered by $\subseteq$

Lemma

For any join semilattice A, $\mathrm{Filt}_e(A)$ is a sublocale of $\mathrm{Up}(A)$.

Proof.

Exact filters are closed under intersection. If U is an up set and E is an exact filter, then $U \to E$ is an exact filter. [This proof closely mimics the proof that $\mathrm{Filt}(L)$ is a sublocale.] □

Essentially due to Bruns & Lakser, '70 (not stated or proved *exactly* this way) in the construction of the injective hulls of semilattices.

Exact filters

Theorem

In any frame,

$$U[S(L)] = \mathrm{Filt}_e(L).$$

Hence $S_c(L)$ is a frame isomorphic to $\mathrm{Filt}_e(L)$, sitting inside the coframe $S(L)$.

Exact filters

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In any frame,

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So

$$\begin{array}{ccc}
\mathrm{Up}(L) & \underset{U}{\overset{J}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} & \mathrm{S}(L) \\
\uparrow \subseteq & & \uparrow \subseteq \\
\mathrm{Filt}_e(L) = U[S(L)] & \underset{\longleftarrow}{\overset{\longrightarrow}{\equiv}} & \mathrm{S}_c(L) = J[\mathrm{Up}(L)]
\end{array}$$

Fitted filters (a similar, but not as nice story)

Definition

For $U \in \mathrm{Up}(A)$, define $\varphi_U \colon A \to A$

$$\varphi_U(x) = \bigvee_{a \in U} (a \to x)$$

Say that filter $F \subseteq L$ is a fitted filter if and only if

- For all $b \in A$,
  - if $\forall x \in A, \varphi_F(x) \leq x$ implies $b \to x \leq x$,
  - then $b \in F$.

Lemma

Fitted subsets of a frame L form a sublocale of Up(L). [Call it $\mathrm{Filt}_f(L)$.]

Fitted filters

For a frame L, consider the maps $M{:}\mathrm{Up}(L)\to S(L)^\partial$, $V{:}S(L)^\partial\to\mathrm{Up}(L)$

$$M(A) = \bigcap_{a\in A} o(a)$$
$$V(S) = \{a \in L \mid S \subseteq o(a)\}$$

Fitted filters

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In the interest of time:

$$\begin{array}{ccc}
\mathrm{Up}(L) & \underset{V}{\overset{M}{\rightleftarrows}} \; \bot & \mathrm{S}(L)^\partial \\
\uparrow \subseteq & & \uparrow \subseteq \\
\mathrm{Filt}_f(L) = V[S(L)^\partial] & \rightleftarrows \; \equiv & \mathrm{S}_o(L)^\partial = M[\mathrm{Up}(L)]
\end{array}$$

So $\mathrm{S}_o(L)$ is a coframe.

Some other consequences

- $S_c(L)$ and $S_o(L)^\partial$ correspond to special filters on L
- If L is fit, then $\mathrm{Filt}_f(L) \simeq^\partial \mathsf{S}(L)$
- If L is scattered, the $\mathrm{Filt}_e(L) \simeq S(L)$
- If L is subfit, then $\mathrm{Filt}_e(L)$ is Boolean
- If L is subfit, then $L \to \mathrm{Filt}_e(L)$ by $a \mapsto \{x \mid x \vee a = 1\}$ is the essential extension of L

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Thank you