The coproduct of frames as encoding d-frame structure

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2. D-frames
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Bitopological spaces

Some topologies naturally arise as the join of two other ones.

- The Euclidian topology on $\mathbb{R}$ is generated by the lower open intervals $\{(-\infty, r) : r \in \mathbb{R}\}$ and the upper ones $\{(r, \infty) : r \in \mathbb{R}\}$.

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- For a Priestley space $(X, \leq)$ the topology is the join of two spectral topologies: the ones of open upsets and open downsets.
- The Vietoris hyperspace VX of a compact Hausdorff space X has as underlying set the closed subsets $\{X \backslash U : U \in \Omega(X)\}$. The topology is the join of the *upper* and *lower* topologies, with bases:
  - $\Box U = \{C \in VX : C \subseteq U\}$.
  - $\Diamond U = \{C \in VX : C \cap U \neq \emptyset\}$.

  Where U varies over $\Omega(X)$.

Bitopological spaces

A *bitopological space* is a structure (X, τ^+, τ^-) where X is a set and τ^+ and τ^- two topologies on it. We call τ^+ the *upper*, or *positive*, topology. We call τ^- the *negative*, or *lower*, topology.

The category **BiTop** has bitopological spaces as objects, *bicontinuous* functions as maps.

D-frames: intuition

D-frames are quadruples $(L^+, L^-, \mathsf{con}, \mathsf{tot})$ where L^+ and L^- are frames, and $\mathsf{con}, \mathsf{tot} \subseteq L^+ \times L^-$; satisfying some axioms. The intuition is:

- L^+ and L^- are the frames of positive and negative **opens** respectively.
- The pairs of opens in **con** are the **disjoint** pairs.
- The pairs of opens in **tot** are the **covering** pairs (i.e. those whose union covers the whole space).

D-frames: two simple examples

- We can set con and tot to be as small as the axioms allow. We may define:
  - $x^+x^- \in \mathsf{con}_m$ if and only if $x^+ = 0^+$ or $x^- = 0^-$.
  - $x^+x^- \in \mathsf{tot}_m$ if and only if $x^+ = 1^+$ or $x^- = 1^-$.

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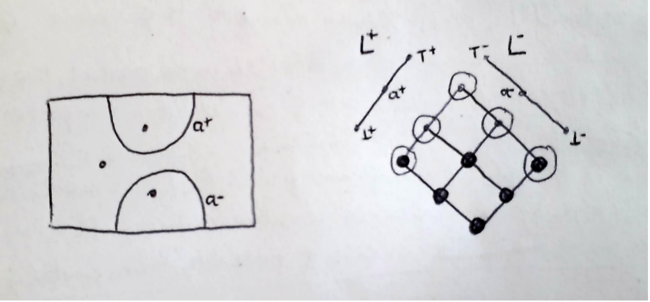
- The following is a bitopological space with its d-frame of opens.



D-frames: axioms

A quadruple $(L^+, L^-, \mathsf{con}, \mathsf{tot})$ where L^+ and L^- are frames and $\mathsf{con}, \mathsf{tot} \subseteq L^+ \times L^-$ is a *d-frame* if the following five axioms hold:

- (con-D) con is closed downwards.
- (con-j) Whenever $\{a_i^+ a^- : i \in I\} \subseteq \mathsf{con}$ also $(\bigvee_i a_i^+, a^-) \in \mathsf{con}$. Similarly for families in L^-.

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- (tot-U) tot is closed upwards.
- (tot-fm) Whenever $\{a_i^+ a^- : i \in F\} \subseteq \mathsf{tot}$ for F finite also $(\bigwedge_i a_i^+, a^-) \in \mathsf{tot}$. Similarly for families in L^-.

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- (Balance). Whenever $a^+ b^- \in \mathsf{con}$ and $a^+ c^- \in \mathsf{tot}$ we have $b^- \leq c^-$. Similarly whenever $b^+ a^- \in \mathsf{con}$ and $c^+ a^- \in \mathsf{tot}$ we have $b^+ \leq c^+$.

D-pseudocomplemets

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For $a^+ \in L^+$, the element $\sim a^+ := \bigvee\{x^- \in L^- : a^+x^- \in \mathsf{con}\}$ is the *d-pseudocomplement* of a^+.

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For any $a^+ \in L^+$:

$$(a^+, \sim a^+) \in \mathsf{con}.$$

The category **dFrm**

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Theorem

There is a Stone-type adjunction $\Omega : \boldsymbol{BiTop} \leftrightarrows \boldsymbol{dFrm}^{op} : \mathsf{pt}$.

The con and tot subsets as subspaces

Abstract subspaces of d-frames are **surjections**. Let us look at all the possible con and tot subsets of $L^+ \times L^-$.

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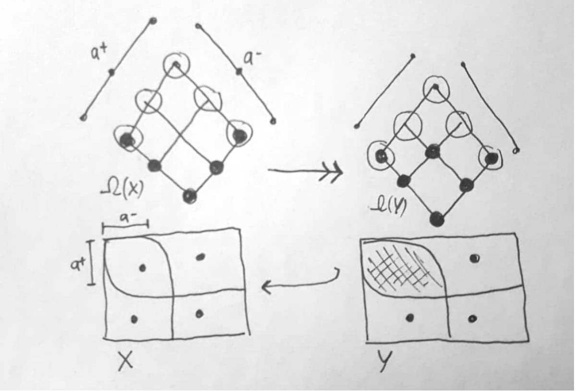
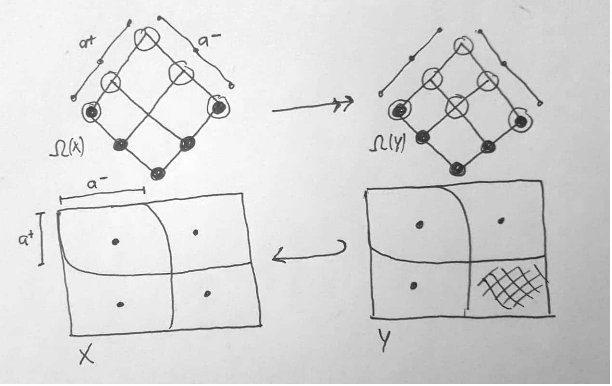
The cons are **closed** subspaces.



The con and tot subsets as subspaces

The tots are **intersections of open** subspaces.



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The con and tot surjections generate all surjections from a d-frame $(L^+, L^- \mathsf{con}_m, \mathsf{tot}_m)$.

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The con and tot surjections generate all surjections from a d-frame $(L^+, L^-\mathsf{con}_m, \mathsf{tot}_m)$.

- We call $\mathsf{con}(L^+ \times L^-)$ the poset of suitable con subsets of $L^+ \times L^-$.
- We call $\mathsf{tot}(L^+ \times L^-)$ the poset of suitable tot subsets of $L^+ \times L^-$.

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The posets $\mathsf{con}(L^+ \times L^-)$ and $\mathsf{tot}(L^+ \times L^-)$ are frames.

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Proposition

The posets $\mathsf{con}(L^+{\times}L^-)$ and $\mathsf{tot}(L^+ \times L^-)$ are frames. They form sublocales of $\mathcal{D}(L^+{\times}L^-)$ and $\mathcal{U}(L^+{\times}L^-)$ respectively.

Galois connections

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$$\frac{\dfrac{x^+ \leq {\sim}x^-}{x^- \leq {\sim}x^+}}{x^+ x^- \in \mathsf{con}}$$

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Lemma

There is an order isomorphism $\mathsf{con}(L^+ \times L^-) \cong \mathsf{Gal}(L^+, L^-)$.

The con subsets in the coproduct

The frame of Galois connections satisfies an important universal property in **Frm**.

Theorem (Wigner, 1979)

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Corollary

There is a frame isomorphism $L^+ \oplus L^- \cong \mathsf{con}(L^+ \times L^-)$.

Since L^+, L^- are subframes of $L^+ \oplus L^-$, for simplicity we will assume notationally that $L^+, L^- \subseteq L^+ \oplus L^-$.

The tot subsets in the coproduct

Consider the subframe of $\mathsf{Filt}(L^+ \oplus L^-)$ generated by $\{{\uparrow}x^+ : x^+ \in L^+\} \cup \{{\uparrow}x^- : x^- \in L^-\}$.

The tot subsets in the coproduct

Consider the subframe of $\mathsf{Filt}(L^+ \oplus L^-)$ generated by $\{\uparrow x^+ : x^+ \in L^+\} \cup \{\uparrow x^- : x^- \in L^-\}$. We denote this as $\mathsf{filt}(L^+ \oplus L^-)$ and call its elements *finitary filters*.

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Proposition

The following is an order isomorphism:

$$i : \mathsf{tot}(L^+ \times L^-) \cong \mathsf{filt}(L^+ \oplus L^-)$$

$$i : t \mapsto \bigvee \{\uparrow x^+ \cap \uparrow x^- : x^+ x^- \in t\}.$$

A d-frame of cons and tots

For any frame L, there is a naturally occurring Galois connection:

$$\uparrow : L \leftrightarrows \mathsf{Filt}(L) : \bigwedge$$

But Galois connections are con subsets. Then we may define a con subset $\mathsf{Con} \subseteq (L^{+} \oplus L^{-}) \times (\mathsf{filt}(L^{+} \oplus L^{-}))$ as follows:

$$\dfrac{\dfrac{xF \in \mathsf{Con}}{F \subseteq {\uparrow}x}}{x \leq \bigwedge F}$$

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Define also Tot, as:

$$\frac{xF \in \mathsf{Tot}}{{\uparrow} x \subseteq F}$$

An assembly-like universal property

Proposition

The structure $(L^+ \oplus L^-, \mathsf{filt}(L^+ \oplus L^-), \mathsf{Con}, \mathsf{Tot})$ *is a d-frame.*

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What is special about this d-frame?

Consider the **free frame** from **generators** $L^+ \oplus L^-, \mathsf{filt}(L^+ \oplus L^-)$ and **relations** $\mathsf{Con}, \mathsf{Tot}$. We call this frame the *finitary assembly* of $L^+ \oplus L^-$ and denote it as $\mathsf{cong}(L^+ \oplus L^-)$.

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For any frame L its *assembly* $\mathsf{Cong}(L)$ is a frame.

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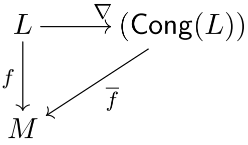
For any frame L its *assembly* $\mathsf{Cong}(L)$ is a frame. There is an embedding $\nabla : L \to \mathsf{Cong}(L)$ such that $\nabla(x)$ is **complemented** for all $x \in L$. This is universal in **Frm** with this property (Joyal and Tierney, 1984).



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Proposition

Consider the canonical embedding of generators in the free frame:

$$[-] : L^+ \oplus L^- \to \mathsf{cong}(L^+ \oplus L^-).$$

This is such that $[x]$ is complemented for every finitary $x \in L^+ \oplus L^-$. The map is universal in ***Frm*** *with this property.*

Characterizations of the finitary assembly

Theorem

The following are all isomorphic frames.

- *The finitary assembly* $\mathsf{cong}(L^+ \oplus L^-)$.
- *The subframe of* $\mathsf{Cong}(L^+ \oplus L^-)$ *generated by* $\{\Delta(x) : x \in L^+ \cup L^-\} \cup \{\nabla(x) : x \in L^+ \cup L^-\}$.
- *The* $\wedge$*-subsemilattice of* $\mathsf{Cong}(L^+ \oplus L^-)$ *generated by the congruences of extremal epis from* $(L^+, L^-, L^+ \oplus L^-)$ *in* ***BiFrm***.
- *The frame of congruences on* $L^+ \oplus L^-$ *of the form* $\bigvee_i \Delta(x_i^+ \vee x_i^-) \cap \nabla(y_i^+ \wedge y_i^-)$.

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Stone spaces