Partial frames

Anneliese Schauerte and John Frith

University of Cape Town

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Partial frames: basic ideas

- A meet-semilattice is a partially ordered set with a top, 1, and bottom, 0, in which all finite meets exist.
- A partial frame, or $\mathcal{S}$-frame, is a meet-semilattice in which certain joins exist and finite meets distribute over these joins.
- We specify the collections whose joins should exist by means of a *selection function*, denoted by $\mathcal{S}$.
- Once a selection function has been chosen, we speak informally of the collections it picks as the *designated* subsets.

Axioms for a selection function

Definition

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(S3) If $G \in \mathcal{S}A$ and, for all $x \in G$, $x = \bigvee H_x$ for some $H_x \in \mathcal{S}A$, then

$$\bigcup_{x \in G} H_x \in \mathcal{S}A.$$

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(S4) For any meet-semilattice map $f : A \to B$,

$$\mathcal{S}(f[A]) = \{f[G] : G \in \mathcal{S}A\} \subseteq \mathcal{S}B.$$

$\mathcal{S}$-frames

Definition

Let $\mathcal{S}$ be a selection function.

1. An $\mathcal{S}$-frame, L, is a meet-semilattice that satisfies the following two conditions:

   (a) For all $G \in \mathcal{S}L$, G has a join in L (i.e. $\bigvee G$ exists).

   (b) For all $x \in L$, for all $G \in \mathcal{S}L$, $x \wedge \bigvee G = \bigvee_{y \in G} x \wedge y$.

2. Let L and M be $\mathcal{S}$-frames. An $\mathcal{S}$-frame map $f : L \to M$ is a meet-semilattice map such that, for all $G \in \mathcal{S}L$, $f(\bigvee G) = \bigvee_{y \in G} f(y)$.

3. $\mathcal{S}\mathbf{Frm}$ is the category of $\mathcal{S}$-frames as objects and $\mathcal{S}$-frame maps as morphisms.

Example

We give several selection functions, together with their corresponding categories of $\mathcal{S}$-frames. Throughout, A is an arbitrary meet-semilattice.

1. $\mathcal{S}A = \{\{x\} : x \in A\}$. $\mathcal{S}$**Frm** is just the category of meet-semilattices.
2. $\mathcal{S}A = \{G \subseteq A : G \text{ is finite}\}$. $\mathcal{S}$**Frm** is the category of bounded distributive lattices.
3. $\mathcal{S}A = \{G \subseteq A : G \text{ is countable}\}$. $\mathcal{S}$**Frm** is the category of σ-frames.
4. $\mathcal{S}A = \{G \subseteq A : \mathrm{card}(G) < \kappa\}$, where $\mathrm{card}(G)$ denotes the cardinality of G and κ is a regular cardinal. $\mathcal{S}$**Frm** is the category of κ-frames.
5. $\mathcal{S}A = \mathcal{P}A$, the power set of A. $\mathcal{S}$**Frm** is the category of frames.

Algebraic ideas and constructions

Algebraic ideas and constructions

Definition

Let L be an $\mathcal{S}$-frame. We call $\theta \subseteq L \times L$ an $\mathcal{S}$-congruence on L if it satisfies the following:

(1) θ is an equivalence relation.

(2) $(a, b), (c, d) \in \theta$ implies that $(a \wedge c, b \wedge d) \in \theta$.

(3) If $\{(a_\alpha, b_\alpha) : \alpha \in \mathcal{A}\} \subseteq \theta$ and $\{a_\alpha : \alpha \in \mathcal{A}\}$ and $\{b_\alpha : \alpha \in \mathcal{A}\}$ are designated subsets of L, then $(\bigvee_{\alpha \in \mathcal{A}} a_\alpha, \bigvee_{\alpha \in \mathcal{A}} b_\alpha) \in \theta$.

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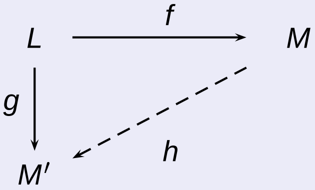
Definition

Let $f : L \to M$ be an $\mathcal{S}$-frame map. We define the kernel of f, by

$$\ker f = \{(x, y) \in L \times L : f(x) = f(y)\}.$$

Lemma

- Let $f : L \to M$ and $g : L \to M'$ be $\mathcal{S}$-frame maps between $\mathcal{S}$-frames with f onto. If $\ker f \subseteq \ker g$, there exists a unique $\mathcal{S}$-frame map $h : M \to M'$ such that $h \circ f = g$, that is, making the following diagram commute:



- $\mathcal{S}$-congruences correspond precisely to kernels and to quotients.

A feature distinguishing partial frames from full frames is the fact that every frame map has a right adjoint: the right adjoint of $f : L \to M$ is a function r from M to L, given by $r(a) = \bigvee\{x \in L : f(x) \leqslant a\}$. This is not so for $\mathcal{S}$-frame maps.

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The following data are equivalent on an $\mathcal{S}$-frame L:

- An $\mathcal{S}$-frame map f from L onto M has a right adjoint.
- An $\mathcal{S}$-congruence has the property that every equivalence class has a largest member.
- $k : L \to L$ is a nucleus.

Generators and relations

Begin with a meet-semilattice A. For each $a \in A$, specify $C(a)$, a collection of designated subsets, each contained in $\downarrow a$.

Aim:
Obtain an $\mathcal{S}$-frame $\mathbb{F}_{\mathcal{S}}(A, C)$ and a meet-semilattice map $f : A \to \mathbb{F}_{\mathcal{S}}(A, C)$ such that $f(a) = \bigvee\{f(s) : s \in S\}$, all $S \in C(a)$, with an appropriate universal property.

Method of construction:
(1) First form the free $\mathcal{S}$-frame over A; this is given by the collection of $\mathcal{S}$-generated downsets of A.
(2) Generate an $\mathcal{S}$-congruence by $\{(\downarrow a, \downarrow S) : a \in A, S \in C(a)\}$ and factor out by this.

Examples

1. Let A be a bounded distributive lattice. For $a \in A$, let $C(a) = \{S \subseteq A : S \text{ is finite and } \bigvee S = a\}$. The free $\mathcal{S}$-frame over the bounded distributive lattice A is given by $A \to \mathbb{F}_{\mathcal{S}}(A, C)$.
2. Adjoining complements freely.
3. Constructing coproducts.

The least dense quotient and the role of Booleanness

In a frame N, the pseudocomplement of a is $a^* = \bigvee\{t \in N \mid t \wedge a = 0\}$.

In a partial frame, pseudocomplements might not exist. Here is an example that is very simple, but will be of repeated use to us:

Example

Let L consist of all countable subsets of $\mathbb{R}$ with top element $\mathbb{R}$. This is a σ-frame, with finite intersections and countable unions.

It is not a frame:

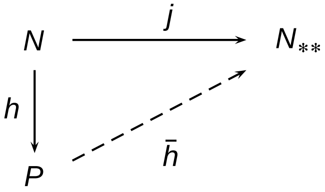
$\{1\} \wedge \bigvee\{\{x\} : x \text{ irrational}\} = \{1\} \wedge \mathbb{R} = \{1\}$ whereas

$\bigvee\{\{1\} \wedge \{x\} : x \text{ irrational}\} = \emptyset$.

The element $\{1\}$ clearly has no pseudocomplement.

For a frame N, the nucleus $j : N \to N$ given by $j(a) = a^{**}$ produces the quotient, N_{**}, of N with two attractive properties:

1. The frame map $j : N \to N_{**}$ is the least dense quotient of N. This means that for any dense, onto frame map $h : N \to P$ where P is a frame, there exists a unique frame map $\bar{h} : P \to N_{**}$ such that $\bar{h} \circ h = j$.



2. The frame map $j : N \to N_{**}$ is the unique dense Boolean quotient of N.

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Definition

Let L be an $\mathcal{S}$-frame.

1. For each $x \in L$ define $P_x = \{t \in L : t \wedge x = 0\}$.
2. Define $\pi_L = \{(x, y) \in L \times L : P_x = P_y\}$.

Proposition

1. The relation π_L is an $\mathcal{S}$-congruence on L, called the Madden congruence of L.
2. The quotient map $p_L : L \to L/\pi_L$ is indeed the least dense quotient of L.

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Answer: No.

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Yes: let $Q_x = \{s \in L \mid s \wedge t = 0 \text{ for all } t \in P_x\}$.

Definition

For L an $\mathcal{S}$-frame, let $\mathcal{Q}L = \{Q_x : x \in L\}$ and let $q_L : L \to \mathcal{Q}L$ be defined by $q_L(x) = Q_x$.

This provides an alternative presentation of the Madden quotient:

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Proposition

The poset $(\mathcal{Q}L, \subseteq)$ is order isomorphic to L/π_L.

Question: Which partial frames do have a Boolean least dense quotient?
Answer: Those which satisfy the condition that, for all $x \in L$, there exists $y \in L$ with $P_x = Q_y$.

The Madden quotient of L and the Booleanization of the free frame over L

The free frame on the $\mathcal{S}$-frame L is $\mathcal{H}_{\mathcal{S}}L$, the frame of $\mathcal{S}$-ideals of L, those ideals closed under taking joins of designated subsets.
For any $x \in L$, $\downarrow x = \{t \in L : t \leqslant x\} \in \mathcal{H}_{\mathcal{S}}L$.

Lemma

Let L be an $\mathcal{S}$-frame, $x \in L$.

1. Both P_x and Q_x are $\mathcal{S}$-ideals of L.
2. In $\mathcal{H}_{\mathcal{S}}L$, $(\downarrow x)^* = P_x$.
3. In $\mathcal{H}_{\mathcal{S}}L$, $(\downarrow x)^{**} = Q_x$.

Let $j : \mathcal{H}_{\mathcal{S}}L \to (\mathcal{H}_{\mathcal{S}}L)_{**}$ be the Booleanization of $\mathcal{H}_{\mathcal{S}}L$, that is, $j(I) = I^{**}$.

Corollary

Let L be an $\mathcal{S}$-frame. Then $\mathcal{Q}L$ is a sub $\mathcal{S}$-frame of $(\mathcal{H}_{\mathcal{S}}L)_{**}$, which makes the following diagram commute. (Here "inc" refers to the inclusion map.)

$$\begin{array}{ccc} L & \xrightarrow{\;q_L\;} & \mathcal{Q}L \\ {\scriptstyle\downarrow}\big\downarrow & & \big\downarrow{\scriptstyle \text{inc}} \\ \mathcal{H}_{\mathcal{S}}L & \xrightarrow{\;j\;} & (\mathcal{H}_{\mathcal{S}}L)_{**} \end{array}$$

Moreover, the image of L under $j\circ\downarrow$ is $\mathcal{Q}L$.

We now have three ways of viewing the Madden quotient:

1. $p_L : L \to L/\pi_L$
2. $q_L : L \to \mathcal{Q}L$
3. $j \circ \downarrow : L \to \mathcal{Q}L$ where $\mathcal{Q}L$ is viewed as a sub $\mathcal{S}$-frame of $(\mathcal{H}_{\mathcal{S}}L)_{**}$.

Skeletal maps provide a reflection

We call a partial frame d-reduced if it is isomorphic to its Madden quotient.
Frame version: Boolean frames.

We call an $\mathcal{S}$-frame map $f : L \to M$ skeletal if $f[Q_x] \subseteq Q_{f(x)}$ for each $x \in L$.
Frame version: $f(x^{**}) \leqslant f(x)^{**}$.

We note that f is skeletal iff $(f \times f)[\pi_L] \subseteq \pi_M$.

We denote the category of $\mathcal{S}$-frames with skeletal maps by $\mathcal{S}\mathbf{Frm}_{Sk}$.

Proposition

Suppose that $f : L \to M$ is a skeletal $\mathcal{S}$-frame map. Define $\bar{f} : L/\pi_L \to M/\pi_M$ by $\bar{f}([x]) = [f(x)]$. Then $\bar{f}$ is an $\mathcal{S}$-frame map which makes the following diagram commute:

$$\begin{array}{ccc} L & \xrightarrow{\quad f \quad} & M \\ {\scriptstyle p_L}\downarrow & & \downarrow{\scriptstyle p_M} \\ L/\pi_L & \xrightarrow{\quad \bar{f} \quad} & M/\pi_M \end{array}$$

Conversely, if there is an $\mathcal{S}$-frame map g such that $g \circ p_L = p_M \circ f$ then f is skeletal.

Corollary

The full subcategory of $\mathcal{S}\mathbf{Frm}_{Sk}$ consisting of the d-reduced objects is reflective in $\mathcal{S}\mathbf{Frm}_{Sk}$.

Use the diagram:

$$\begin{array}{ccc} L & \xrightarrow{\quad f \quad} & M \\ {\scriptstyle p_L}\big\downarrow & & \big\downarrow{\scriptstyle p_M}\ \cong \\ L/\pi_L & \overset{\bar{f}}{\dashrightarrow} & M/\pi_M \end{array}$$

Here we use the fact that taking Madden quotients is idempotent.

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Answer: Yes.

For any pair of elements a and b of an $\mathcal{S}$-frame L, define $H_{(a,b)}$ to be $\{z \in L : z \wedge a \leqslant b\}$.

One can then prove that $H_{(a,b)}$ is in fact $\downarrow a \rightarrow \downarrow b$ in $\mathcal{H}_{\mathcal{S}}L$ and the usual identities for a Heyting arrow hold.

De Morgan partial frames

We recall that a de Morgan frame is one in which $x^* \vee x^{**} = 1$ for all x.

Definition

We call an $\mathcal{S}$-frame L de Morgan if, for any $x \in L$, $P_x \vee Q_x = \downarrow 1$ in $\mathcal{H}_\mathcal{S} L$.

For frames, the de Morgan condition is equivalent to (for all a, b):

$$(a \wedge b)^* = a^* \vee b^* \text{ or } (a \vee b)^{**} = a^{**} \vee b^{**}$$

Proposition

If L is a de Morgan $\mathcal{S}$-frame then, for any $a, b \in L$, $P_a \vee P_b = P_{a \wedge b}$ and $Q_a \vee Q_b = Q_{a \vee b}$ in $\mathcal{H}_\mathcal{S} L$.

The converse of the proposition above is false; in fact, it is possible to have $P_a \vee P_b = P_{a \wedge b}$ and $Q_a \vee Q_b = Q_{a \vee b}$ in $\mathcal{H}_{\mathcal{S}}L$ for all a, b but $P_x \vee Q_x \neq \downarrow 1$ for some x:

Example

Let L be the collection of countable subsets of $\mathbb{R}$ with top element $\mathbb{R}$.
For a (non-empty) countable subset $X \subseteq \mathbb{R}$, we have:
$P_X = \{A \subseteq \mathbb{R} : A$ is countable and $A \cap X = \varnothing\}$ and
$Q_X = \downarrow X$.
For such X, $P_X \vee Q_X \neq \downarrow 1$ since this would require two disjoint countable subsets of $\mathbb{R}$ whose union is not countable. So L is not de Morgan.
On the other hand, it is straightforward to check that $P_X \vee P_Y = P_{X \cap Y}$ and $Q_X \vee Q_Y = Q_{X \cup Y}$.

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In fact, it is a frame.

This involves knowing that, for $\theta \in \mathcal{C}_{\mathcal{S}}L$, we have
$\theta = \bigvee\{\nabla_b \wedge \Delta_a : (a, b) \in \theta \text{ and } a \leqslant b\}$.

∇_a is the smallest $\mathcal{S}$-congruence identifying 0 with a.
Δ_b is the smallest $\mathcal{S}$-congruence identifying b with 1.

There are restrictions on $\mathcal{S}$.

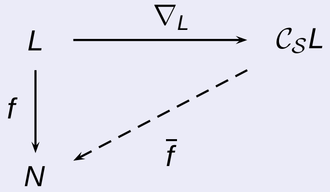
A nice universal property

Theorem

Let $f : L \to N$ be an $\mathcal{S}$-frame map where:

- L is an $\mathcal{S}$-frame and N is a frame and
- the map f has complemented image in N.

Then there is a unique frame map $\bar{f} : \mathcal{C}_{\mathcal{S}}L \to N$ such that $f = \bar{f} \circ \nabla_L$, that is, the following diagram commutes.



And then

Corollary

Taking the $\mathcal{S}$-congruence frame of an $\mathcal{S}$-frame provides a functor $\mathcal{C}_{\mathcal{S}}$ from $\mathcal{S}$-frames to frames.

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Proposition

For L an $\mathcal{S}$-frame, the congruence frame on L, $\mathcal{C}_{\mathcal{S}}L$, is isomorphic to the free frame on $\mathrm{FC}_{\mathcal{S}}L$, the freely complemented $\mathcal{S}$-frame on L. That is

$$\mathcal{C}_{\mathcal{S}}L \cong \mathcal{H}_{\mathcal{S}}\mathrm{FC}_{\mathcal{S}}L$$

Properties of the congruence frame

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It is clear that the congruence frame, $\mathcal{C}_{\mathcal{S}}L$, is generated by its complemented elements and is thus a zero-dimensional, and so completely regular, frame.

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When is $\mathcal{C}_{\mathcal{S}}L$ compact?
Exactly when L is Noetherian, that is, for each $x \in L$, if $x = \bigvee S$ where S is a designated subset of L then $x = s_1 \vee s_2 \vee \ldots \vee s_n$, for $s_1, s_2, \ldots, s_n \in S$.

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When is $\mathcal{C}_{\mathcal{S}}L$ spatial?
Exactly when L/θ is a spatial $\mathcal{S}$-frame for each θ in $\mathcal{C}_{\mathcal{S}}L$.