The spectrum of a localic semiring

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The Zariski spectrum

- The Zariski spectrum of a commutative ring R is a 'space' on which the elements of R behave like functions.
- Each element $f \in R$ gives a basic open $\overline{f}$ corresponding the region on which it is nonzero. Note that:
  - 0 is never nonzero,
  - 1 is always nonzero,
  - If $f + g$ is nonzero somewhere, then either f or g must be nonzero.
  - fg is nonzero precisely where both f and g are nonzero.
- So the Zariski spectrum is given by the frame presentation

$$\langle \overline{f} : f \in R \mid \overline{0} = 0,\ \overline{1} = 1,\ \overline{f+g} \leqslant \overline{f} \vee \overline{g},\ \overline{fg} = \overline{f} \wedge \overline{g} \rangle.$$

- This frame is isomorphic to the frame of radical ideals of R.
- The points of the frame correspond to prime anti-ideals.

Other kinds of spectrum

- There are number of other very similar spectrum constructions.
- They apply to various different localic commutative semirings.

Class of semiring	Spectrum	Opens	Points
Commutative rings	Zariski	Radical ideals	Prime ideals*
Distributive lattices	Stone	Ideals	Prime filters
Commutative C*-algebras	Gelfand	Closed* ideals	Closed prime ideals
Continuous frames (Scott topology)	Hofmann–Lawson	Closed* ideals	Open prime filters

Open prime anti-ideals

- Let R be a localic semiring with zero/unit maps $\varepsilon_0, \varepsilon_1 \colon \mathcal{O}R \to \Omega$ and addition/multiplication maps $\mu_+, \mu_\times \colon \mathcal{O}R \to \mathcal{O}R \oplus \mathcal{O}R$.

- A point in a spectrum is given by an open prime anti-ideal, which we imagine containing the functions that are nonzero at that point.
- These opens should vary continuously over the spectrum.

- An open prime anti-ideal of R fibred over X is an element $u \in X \oplus \mathcal{O}R$ satisfying
    - $(X \oplus \varepsilon_0)(u) = 0$,
    - $(X \oplus \varepsilon_1)(u) = 1$,
    - $(X \oplus \mu_+)(u) \leqslant (X \oplus \iota_1)(u) \vee (X \oplus \iota_2)(u)$,
    - $(X \oplus \mu_\times)(u) = (X \oplus \iota_1)(u) \wedge (X \oplus \iota_2)(u)$,

  where ι_1 and ι_2 are the coproduct injections.

Defining a general notion of spectrum

- We expect there to be an open prime anti-ideal υ fibred over the spectrum $\operatorname{Spec} R$ which acts as if it contains the pairs (x, f) for which $f(x)$ is nonzero.

- In fact, any open prime anti-ideal α fibred over X gives a family of places for elements of R to be nonzero. Since $\operatorname{Spec} R$ contains *all* such places this should define a locale map $X \to \operatorname{Spec} R$.

- Thus, we define a functor $\mathsf{OPAI}_R \colon \mathbf{Frm} \to \mathbf{Set}$ such that $\mathsf{OPAI}_R(X)$ is the set of open prime anti-ideals over X.
- The spectrum of R is the representing object of OPAI_R if it exists. (Here υ is the universal element.)

- Spectra need not exist in general.

Closed* ideals as suplattice homomorphisms

- We expect the opens of the spectrum of R to correspond to some notion of closed* radical ideals in R.

- The closed sublocales of a frame L are in order-reversing bijection with the elements of L.
- At least classically, suplattice homomorphisms from L to $\Omega = \{0, 1\}$ are also in order-reversing bijection with elements of L via the map $h \mapsto h_*(0)$.

- A closed sublocale S then corresponds to a suplattice homomorphism $a \mapsto [\![S \between a]\!]$.

- So we will identify the ideals with certain suplattice homomorphisms from $\mathcal{O}R$ to Ω.

The quantale of ideals

- A suplattice Q equipped with a commutative monoid structure satisfying $a \bigvee_\alpha b_\alpha = \bigvee_\alpha a b_\alpha$ is called a (commutative) quantale.
- The homset $\mathbf{Sup}(\mathcal{O}R, \Omega)$ has the structure of a suplattice.
- The addition and multiplication operations on R then induce two quantale structures on $\mathbf{Sup}(\mathcal{O}R, \Omega)$.
- An ideal is an element a of $\mathbf{Sup}(\mathcal{O}R, \Omega)$ satisfying $0_+ \leqslant a$, $a + a \leqslant a$ and $a \times b \leqslant a$.
- The set $\mathsf{Idl}(R)$ of ideals inherits a quantale structure from the multiplicative quantale structure on $\mathbf{Sup}(\mathcal{O}R, \Omega)$.
- Every quantale Q has a universal quotient $\rho \colon Q \twoheadrightarrow L$ turning it into a frame (where multiplication is meet).
- Applying this to $\mathsf{Idl}(R)$ gives the frame of radical ideals, $\mathsf{Rad}(R)$.

The universal element

- We are hoping that $\mathrm{Rad}(R)$ is (often) the spectrum of R.
- For this we need a universal element $\upsilon \in \mathrm{OPAI}_R(\mathrm{Rad}(R))$.
- Recall that this element represents an open of $\mathrm{Rad}(R) \oplus \mathcal{O}R$ which "contains the pairs (x, f) for which $f(x)$ is nonzero".

- Each 'function' f in R cuts out a cozero set. If R is spatial, we might try build υ as the union $\bigcup_f \llparenthesis f \rrparenthesis \times \pi(f)$, where $\pi(f)$ is an open set of functions which are nonzero wherever f is.

- A subset S of a ring is called saturated if $fg \in S \implies f \in S$.
- We say an open $s \in \mathcal{O}R$ is saturated if $\mu_\times(s) \leqslant \iota_1(s)$.

- We then set $\pi(I) = \bigwedge\{s \text{ saturated} \mid I \between s\} \in \mathcal{O}R$.

Approximable semirings

- We call a localic semiring R approximable if for all $a \in \mathcal{O}R$, $a \leqslant \bigvee_{I \between a} \pi(I)$.

Theorem

Let R be an overt approximable semiring. Then the spectrum of R exists and is isomorphic to $\mathsf{Rad}(R)$. The universal element is

$$\upsilon = \bigvee_{I \in \mathsf{Idl}(R)} \rho(I) \oplus \pi(I).$$

- As a corollary*, the aforementioned examples of spectra exist and are given by their usual constructions.