Singly Generated Quasivarieties and Residuated Structures

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Consider a logic $\vdash$, algebraized by a quasivariety **K** of algebras.

It may happen (as in classical prop. logic) that the derivable inference rules of $\vdash$ are determined by a *single* set of 'truth tables', i.e., by the operation tables of a single member of **K**.

This happens just in case

$$\mathsf{K} = \mathbb{Q}(\boldsymbol{A}) = \mathbb{ISPP}_{\mathbb{U}}(\boldsymbol{A}) \text{ for some } \boldsymbol{A} \in \mathsf{K} \quad \ldots \quad (*).$$

Subtlety: When some $\boldsymbol{A} \in \mathsf{K}$ determines the *finite* rules of $\vdash$, then another $\boldsymbol{B} \in \mathsf{K}$ determines *all* of the rules.

Note: When **K** is a *variety*, it does not suffice (for $(*)$) that $\mathsf{K} = \mathbb{V}(\boldsymbol{A}) = \mathbb{HSP}(\boldsymbol{A})$ for some $\boldsymbol{A} \in \mathsf{K}$.

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Theorem [essentially Maltsev 1966; Łoś & Suszko 1958].
The following conditions on a quasivariety **K** are equivalent.

(1) $\mathsf{K} = \mathbb{Q}(\boldsymbol{A})$ for some $\boldsymbol{A} \in \mathsf{K}$ (i.e., **K** is 'singly generated').

(2) **K** has the *joint embedding property* (JEP), i.e., any two non-trivial members of **K** both embed into some third member.

(3) For every *set* $\mathsf{X} \subseteq \mathsf{K}$, there exists $\boldsymbol{B} \in \mathsf{K}$ s.t. $\mathsf{X} \subseteq \mathbb{IS}(\boldsymbol{B})$.

(4) [a robust '*relevance principle*']:

For any finite set $\Gamma \cup \Delta \cup \{\alpha \approx \beta\}$ of equations, where Γ is satisfiable in a nontrivial member of **K** and involves different variables from $\Delta \cup \{\alpha \approx \beta\}$,

$$\text{if } \mathsf{K} \models (\&(\Gamma \cup \Delta)) \Rightarrow \alpha \approx \beta, \text{ then } \mathsf{K} \models (\&\Delta) \Rightarrow \alpha \approx \beta.$$

Note. (2) is obviously categorical, therefore so are the others.

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Examples. (1) {Boolean algebras} $= \mathbb{Q}(\mathbf{2})$ has the JEP.

(2) So do all subquasivarieties of **HA** := {Heyting algebras}.

(As a feature of *intuitionistic logic*, '**HA** $= \mathbb{GQ}_\omega(\boldsymbol{A})$' is unexpected, but the algebra ***A*** can't be chosen countable [Wroński 1974].)

(3) [algebras of *relevance logic*]:

RA := {relevant algebras} has the JEP [Tokarz 1979], but **DMM** := {De Morgan monoids} does not.

The varieties **RA** and **DMM** algebraize the Anderson-Belnap logic **R** (without 'Ackermann constants') and its conservative expansion $\mathbf{R^t}$ (with the constants ***t***, ***f***).

The literature on **R** emphasizes a more fragile 'relevance principle': if $\vdash_{\mathbf{R}} \alpha \rightarrow \beta$, then α and β have a common variable.

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As the JEP is a relevance principle, failing in **DMM**, our aim is to

describe the (quasi)varieties of De Morgan monoids that have the JEP, or related 'completeness' properties.

(These algebras will be defined later.)

It is profitable to conduct a universal-algebraic analysis first.

An algebra is said to be 0-*generated* if it has a distinguished element and no proper subalgebra.

Fact. In a quasivariety $\mathbf{K}$ with the JEP and a constant symbol,

(1) the nontrivial 0-generated algebras are isomorphic, and

(2) either every algebra has a trivial subalgebra, or no nontrivial algebra does.

(3) when existing, the unique nontrivial 0-generated $\boldsymbol{A} \in \mathbf{K}$ is *relatively simple*, i.e., it has no proper non-identity congruence θ s.t. $\boldsymbol{A}/\theta \in \mathbf{K}$.

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Theorem. If K has the JEP, and if its nontrivial members lack trivial subalgebras, then there's a relatively simple $\boldsymbol{A} \in \mathsf{K}$ such that $\mathbb{ISP}_{\mathbb{U}}(\boldsymbol{A})$ includes every relatively simple algebra $\boldsymbol{B} \in \mathsf{K}$.

(I.e., every such $\boldsymbol{B}$ models the universal theory of $\boldsymbol{A}$.)

In particular, if K is also *relatively semisimple* (i.e., its relatively subdirectly irreducible members are relatively simple), then

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$\exists x_1 \ldots \exists x_n \, \Phi$ (Φ a disjunction of conjunctions of equations).

Note. PSC is hereditary. Also, SC implies PSC.

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The common retract $\boldsymbol{C}$ is finite and either relatively simple or trivial. A nontrivial common retract is unique, up to $\cong$.

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Definition. The variety **DMM** of *De Morgan monoids* $\mathbf{A} = \langle A; \wedge, \vee, \neg, \cdot, e \rangle$ consists of all distributive lattice-ordered commutative monoids with an 'involution' $\neg$, satisfying

$$x \leqslant x^2 := x \cdot x \qquad \neg\neg x \approx x$$

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Theorem. The maximal PSC subquasivarieties of **DMM** are exactly the classes

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where $\mathbf{A}$ is any 0-generated De Morgan monoid.

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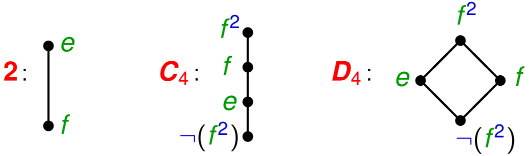

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Theorem. A variety **K** of De Morgan monoids has the JEP iff one of the following (mutually exclusive) conditions holds.

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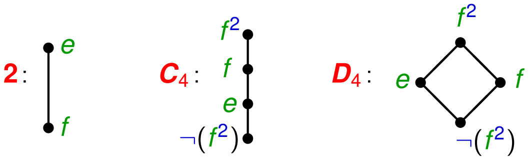

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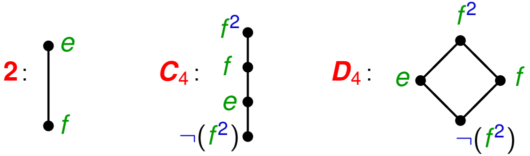

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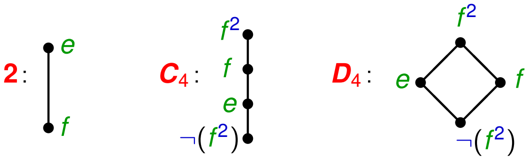

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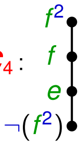

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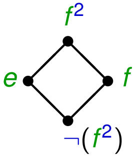

(where $f := \neg e$).

Theorem. A variety **K** of De Morgan monoids has the JEP iff one of the following (mutually exclusive) conditions holds.

(1) **K** is PSC (classified further in the next theorem).

(2) $\mathbf{K} = \mathbb{V}(\boldsymbol{A})$ for a simple De Morgan monoid $\boldsymbol{A}$ that has $\boldsymbol{D}_4$ as a proper subalgebra.

(3) $\mathbf{K} = \mathbb{Q}(\boldsymbol{A})$ for a De Morgan monoid $\boldsymbol{A}$ that has a simple subalgebra $\boldsymbol{B}$, where $\boldsymbol{C}_4$ is a proper subalgebra of $\boldsymbol{B}$.

Ret(**DMM**, **2**) [resp. Ret(**DMM**, $\boldsymbol{D_4}$)] has no nontrivial subvariety, other than $\mathbb{V}(\mathbf{2})$ [resp. $\mathbb{V}(\boldsymbol{D_4})$].

But Ret(**DMM**, $\boldsymbol{C_4}$) has $2^{\aleph_0}$ subvarieties, and it has a *largest* subvariety **M**—which is axiomatized, relative to **DMM**, by

$$e \leqslant f \text{ and } x \leqslant f^2 \text{ and } f^2 \cdot \neg((f \cdot x) \wedge (f \cdot \neg x)) \approx f^2.$$

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(1) **K** is $\mathbb{V}(\mathbf{2})$ or $\mathbb{V}(\boldsymbol{D_4})$.

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Every De Morgan monoid $\boldsymbol{A}$ has a '*negative cone*' $\boldsymbol{A}^- = \langle A^-; \wedge, \vee, \to, e\rangle$ that is a Brouwerian algebra, where $A^- := \{a \in A : a \leqslant e\}$ and $a \to b := e \wedge \neg(a \cdot \neg b)$ for all $a, b \in A^-$.

Conversely, each Brouwerian algebra $\boldsymbol{B}$ has a '*reflection*' $\mathrm{R}(\boldsymbol{B}) \in \mathbf{M}$, illustrated on the right [Meyer 1973].

for $x, y \in B$, $z \in \mathrm{R}(\boldsymbol{B})$:

$x \cdot y = x \wedge y$

$x \cdot y' = (x \to y)'$

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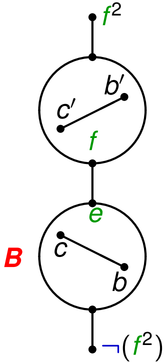

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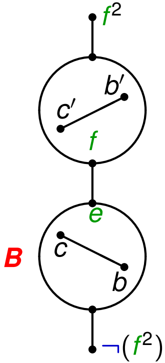

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The lemma reduces our task to that of exhibiting $2^{\aleph_0}$ structurally incomplete (locally finite) varieties of Brouwerian algebras.

Kuznetsov (1975) exhibited a set of $\aleph_0$ finite Brouwerian algebras, distinct subsets of which generate distinct varieties.

His algebras can be modified so that, after a certain extra algebra $\boldsymbol{B}$ is added to the set, distinct subsets *containing* $\boldsymbol{B}$ generate distinct *structurally incomplete* subvarieties (that are locally finite).

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The new algebras are best described via their dual (upper-bounded) posets of prime filters. The corresponding posets of Kuznetsov are modified as in the example below.

For each doubleton $\{x, y\}$ at depth 3, a new element e_{xy} is added, which has no strict lower bound; its set of strict upper bounds is the upward closure of $\{x, y\}$ in the original poset.

The up-sets of the above-right poset form the extra algebra $\boldsymbol{B}$.

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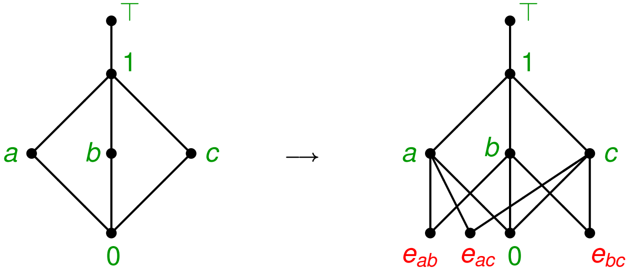

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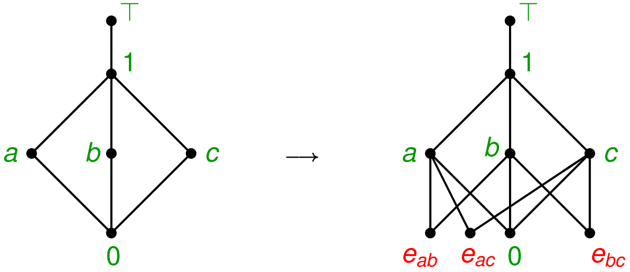

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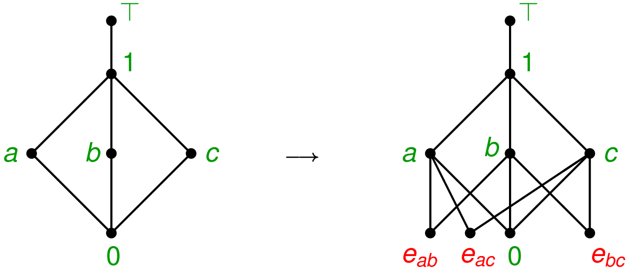

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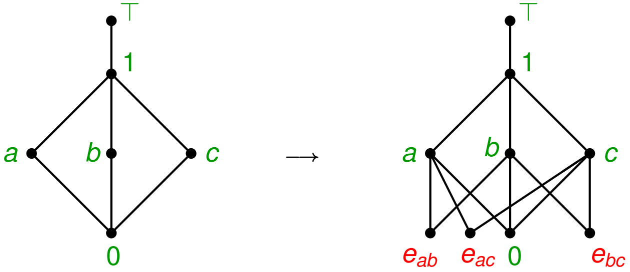

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