A topos for piecewise-linear geometry, and its logic

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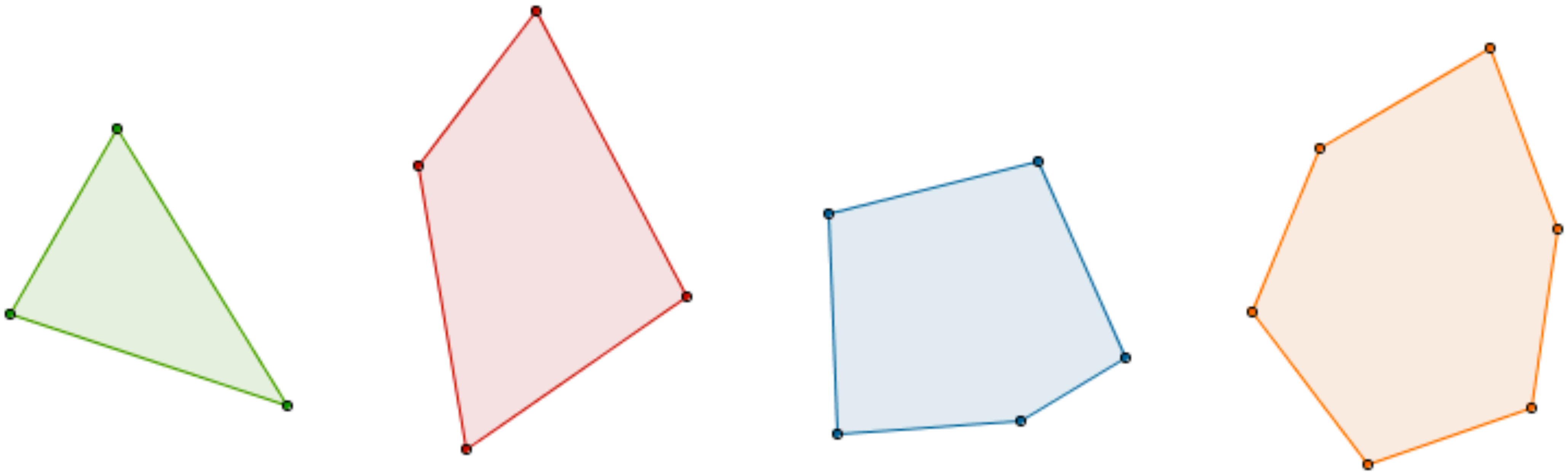
Polytopes are convex hulls of finite sets of points



Some polygons in the plane

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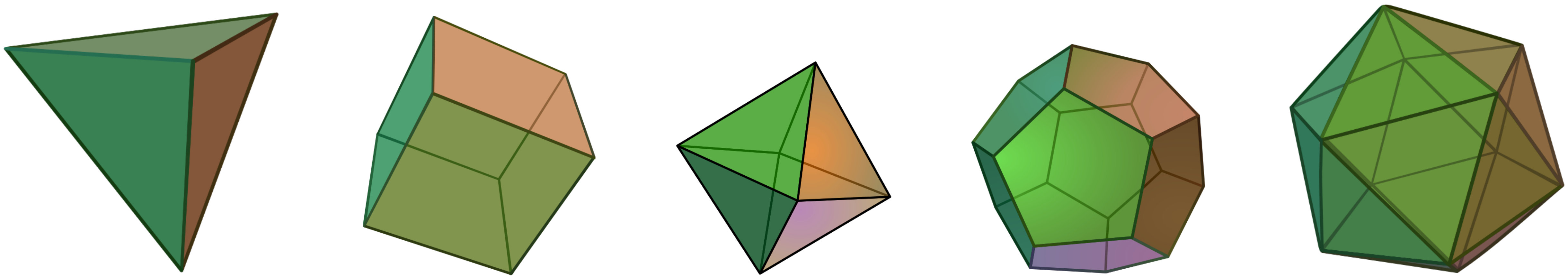
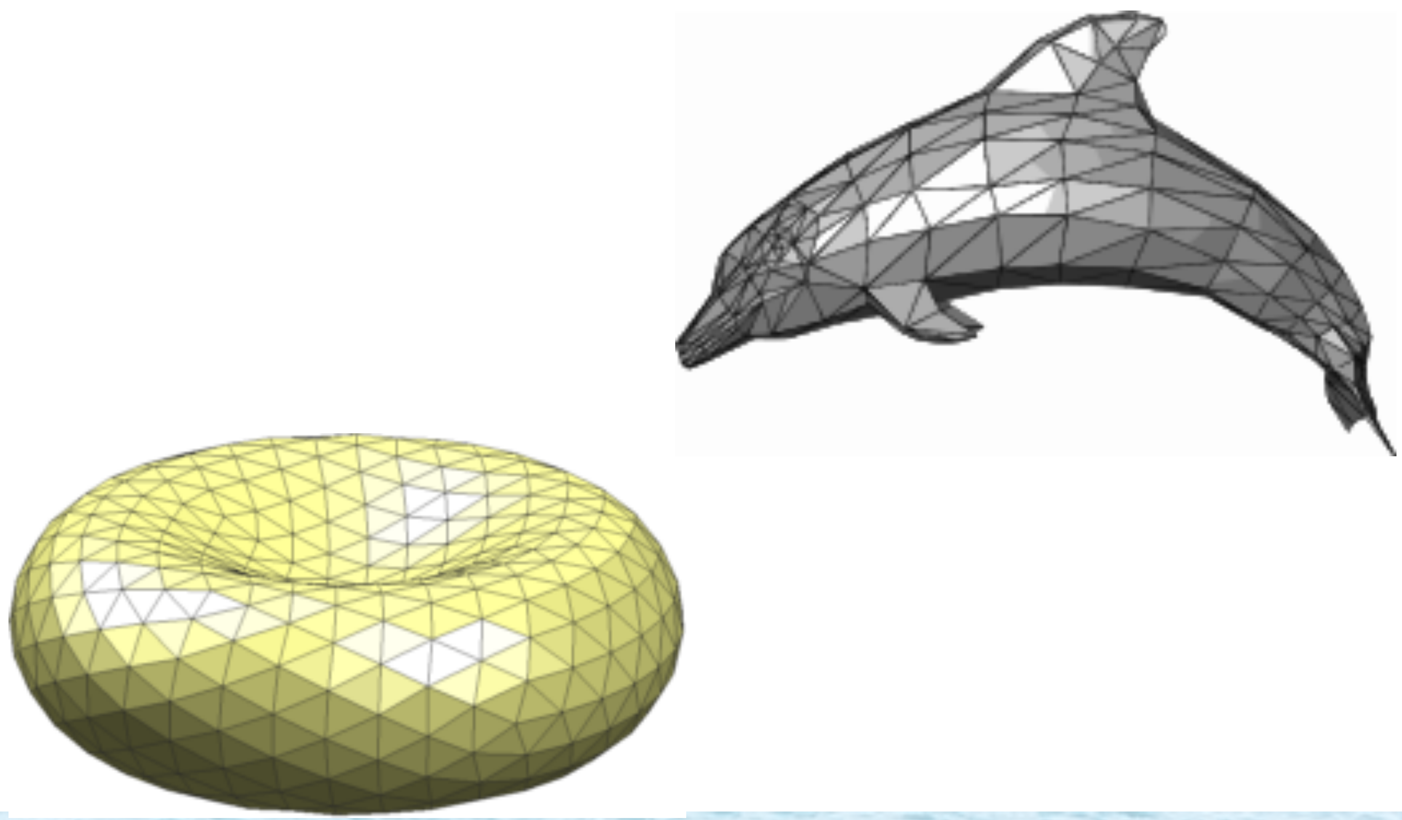


The Platonic solids

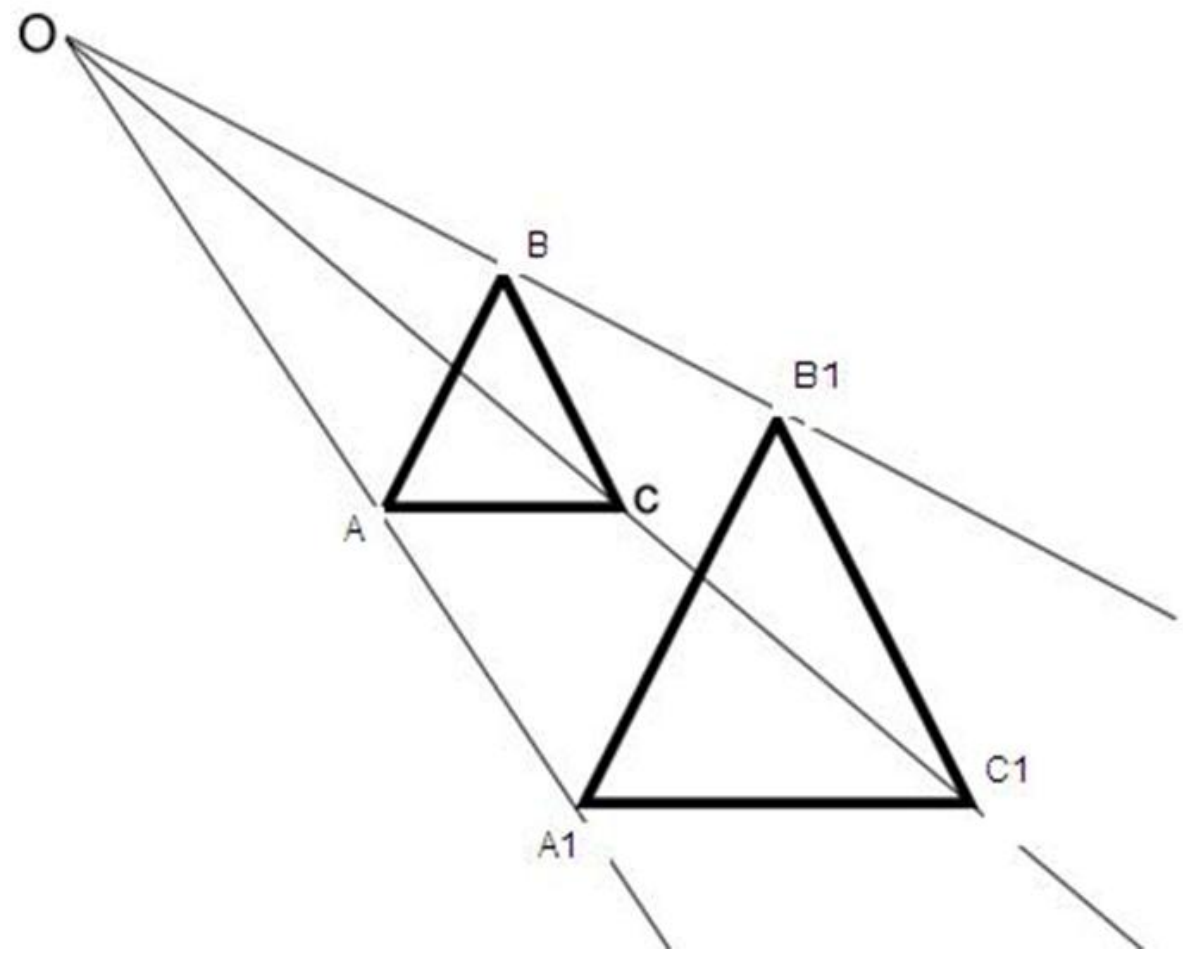
(*After Plato's "Timaeus", ca. 350 BC.*)

Polyhedra are finite unions of polytopes



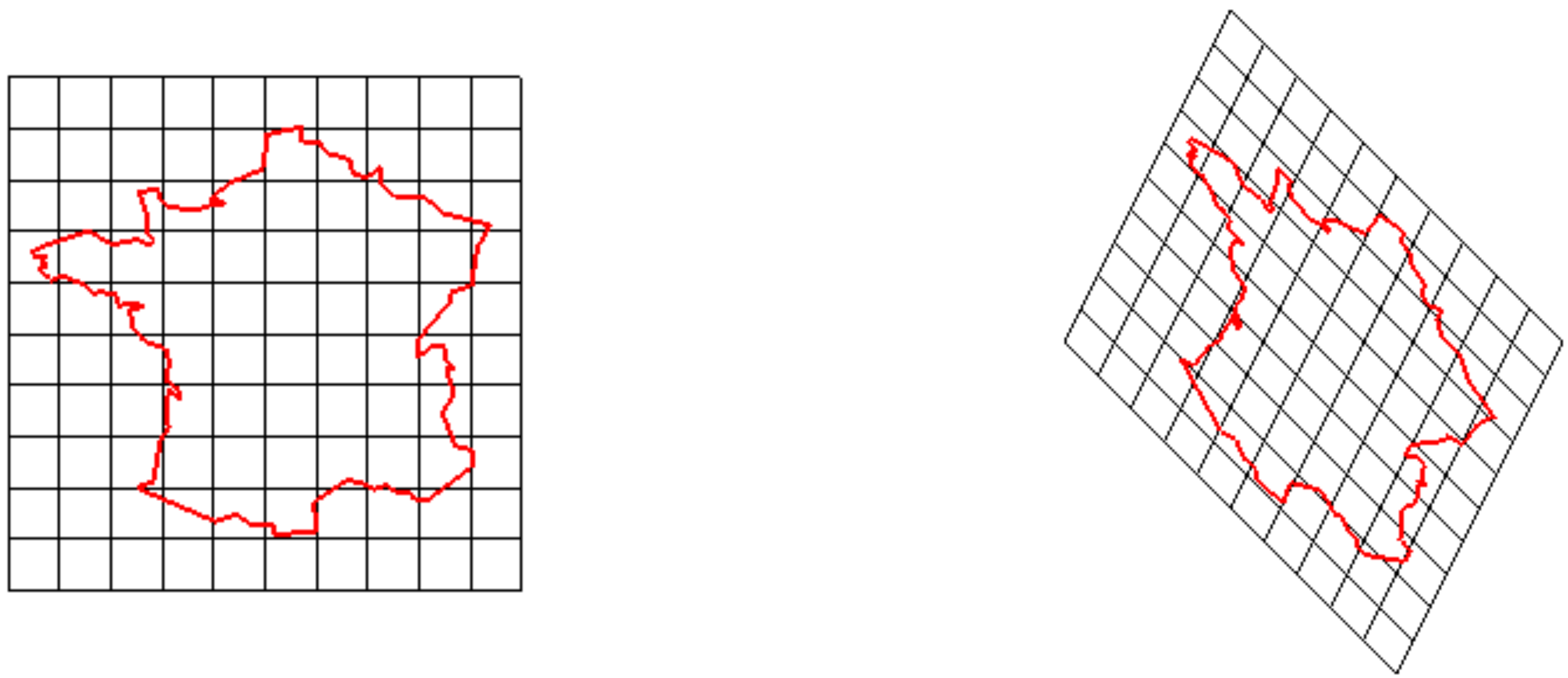
A dolphin playing with a torus

Morphisms between polytopes are affine maps



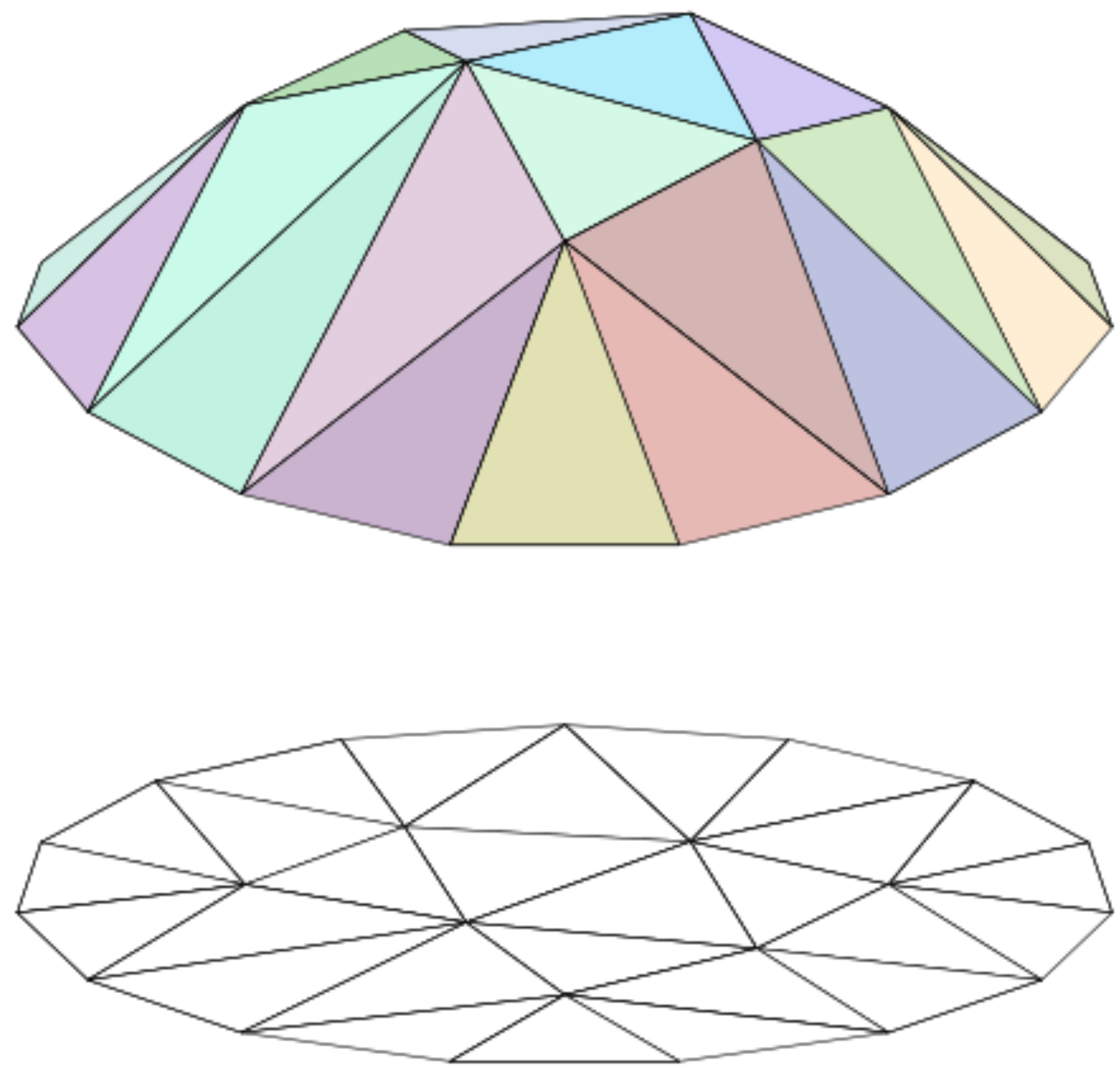
Affine maps are linear transformations with respect to an appropriately chosen origin. They don't preserve distance, angles or orientation, but they do preserve *parallelism* and *convexity*.

Morphisms between polytopes are affine maps



An affine transformation of France

Morphisms between polyhedra are piecewise-affine (PL) maps



Graph of a PL function. PL maps are obtained by gluing together finitely many affine maps, just like polyhedra are obtained gluing together finitely many polytopes. So PL maps are “locally affine maps”.

Two classical categories from geometry

- For $n \in \mathbb{N}$, write $\mathbb{A}^n$ for real n-dimensional affine space.
- C is the *category of polytopes*, with objects polytopes $C \subseteq \mathbb{A}^n$, and morphisms the affine maps.
- P is the *category of polyhedra*, with objects the polyhedra $P \subseteq \mathbb{A}^n$, and morphisms the PL maps.

These categories are "too small". E.g., affine space itself is not a polyhedron; exponentials (function spaces) do not exist; there is no classifier of subobjects; there is no "polyhedron of parts" of a polyhedron (power object); and so on.

We therefore seek a geometrically significant completion.

Taking presheaves

Write $\widehat{\mathsf{C}}$ for the category of *presheaves* on C. Objects are functors

$$\mathsf{C}^{\mathrm{op}} \longrightarrow \mathsf{Set}$$

and morphisms are the natural transformations. Each object c of C gives rise to the *representable* presheaf

$$\mathsf{C}^{\mathrm{op}} \xrightarrow{\mathrm{hom}_{\mathsf{C}}(-,\mathsf{c})} \mathsf{Set}.$$

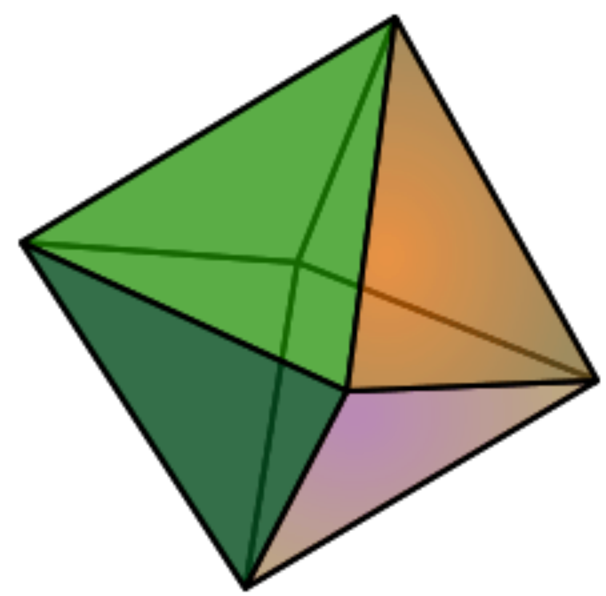
The induced functor

$$\begin{aligned} \mathsf{C} &\xrightarrow{\quad Y \quad} \widehat{\mathsf{C}} \\ \mathsf{c} &\longmapsto \mathrm{hom}_{\mathsf{C}}(-,\mathsf{c}) \end{aligned}$$

is the fully faithful *Yoneda emebedding*.

Presheaves on the category of polytopes (form a topos, and) are "generalised polyhedra". But they are much too general.

Yoneda messes colimits up (as it should).



The Yoneda embedding (is the free colimit completion, and hence) fails to preserve the colimits that exist in the category of polytopes. The octahedron is covered by two tetrahedra, but the presheaf topos does not retain this basic geometric information. (By contrast, the Yoneda embedding preserves all existing limits.)

Gluing after Grothendieck *et al*.

To remedy this, we put a Grothendieck topology on the category of polytopes. It says that the circumstance that a poytope is covered by a *finite* collection of subpolytopes should be recorded by the topos.

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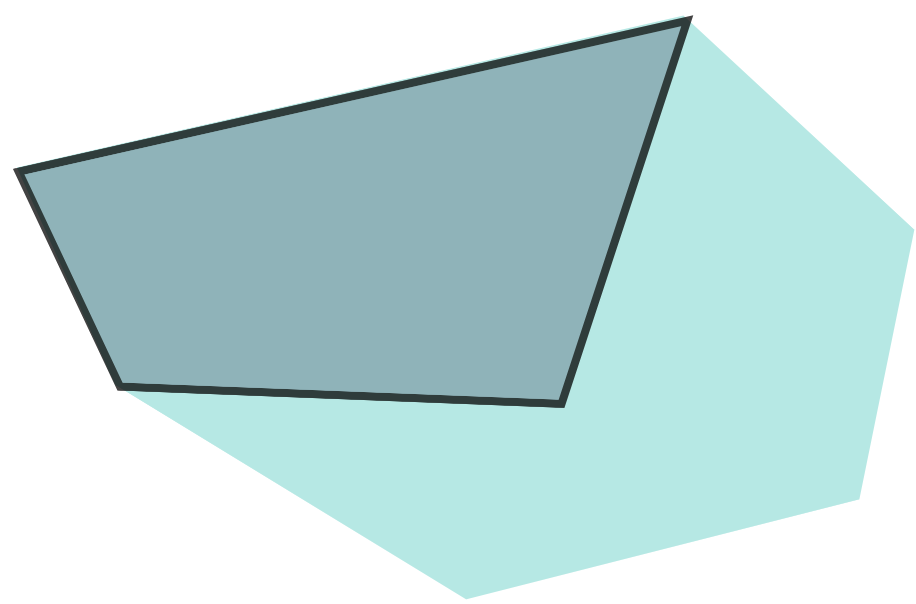

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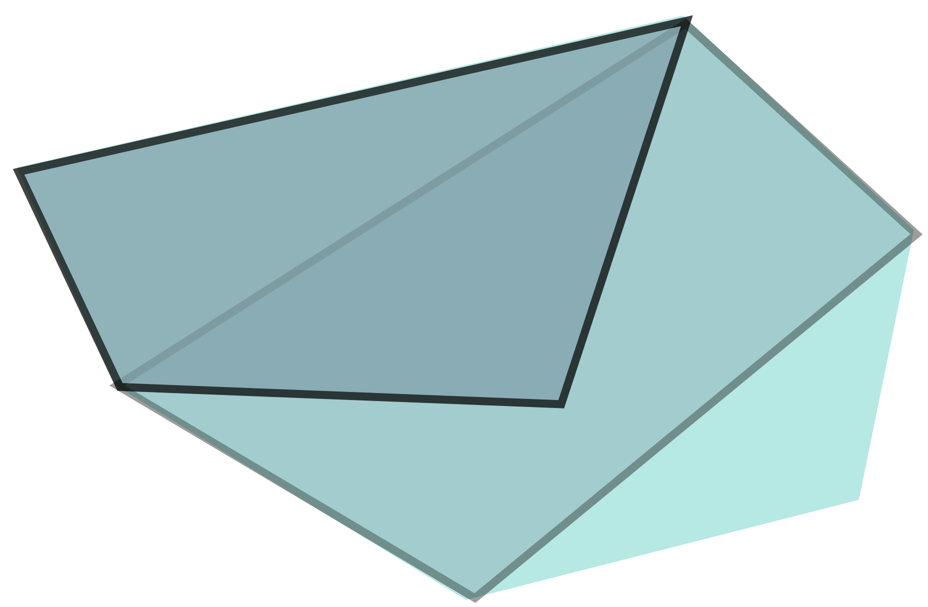

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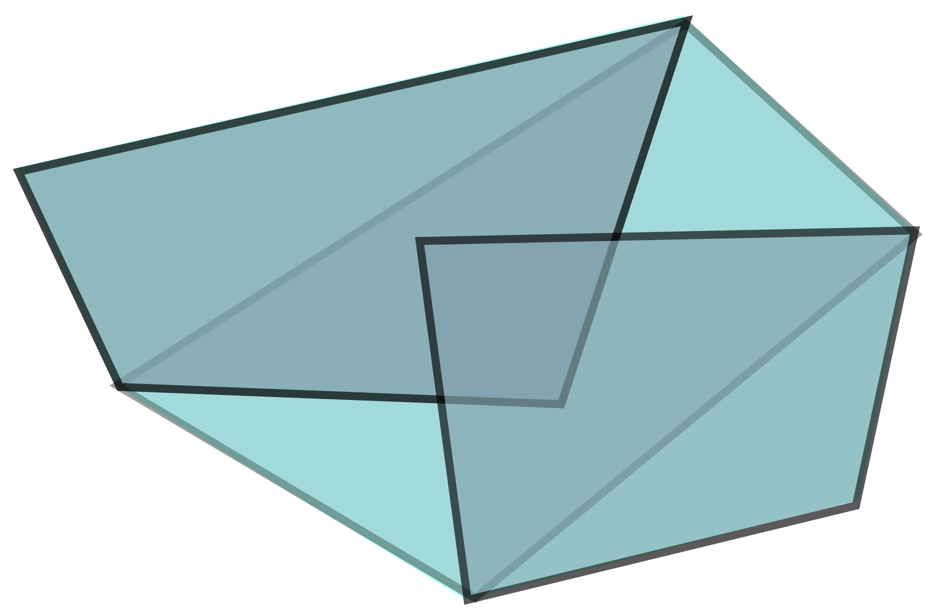

Gluing after Grothendieck — The PL topos

We define J to be the topology on C determined by all *finite* families of *injective* affine maps $C_i \xrightarrow{f_i} \mathsf{D}$ such that their joint image covers D:

$$\mathsf{D} = \bigcup_i f_i[C_i].$$

We call (C, J) the *affine site*.

Definition. The *PL topos* is the topos $\mathscr{P}$ of sheaves on the affine site (C, J).

The topology J is not subcanonical: a representable presheaf need not be a sheaf.

We could start from the polyhedral category instead.

Gluing after Grothendieck — The PL topos, again

We define J^* to be the topology on P determined by all *finite* families of *injective* PL maps $\mathsf{C}_i \xrightarrow{f_i} \mathsf{D}$ such that their joint image covers D:

$$\mathsf{D} = \bigcup_i f_i[\mathsf{C}_i].$$

We call $(\mathsf{P}, \mathsf{J}^*)$ the *PL site*; write $\mathscr{P}^*$ for the resulting sheaf topos. There is a non-full, faithful inclusion of sites $\mathrm{I}\colon (\mathsf{C}, \mathsf{J}) \longrightarrow (\mathsf{P}, \mathsf{J}^*)$.

Theorem. *The geometric morphism $\mathscr{P}^* \longrightarrow \mathscr{P}$ induced by I is an equivalence of categories.*

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There is just one PL topos $\mathscr{P}$, but it can be defined either on the affine site or on the PL site. The PL topos is a very good model of the Lawvere-Menni theory of Axiomatic Cohesion. We turn to logic.

What is the theory classified by the PL topos?

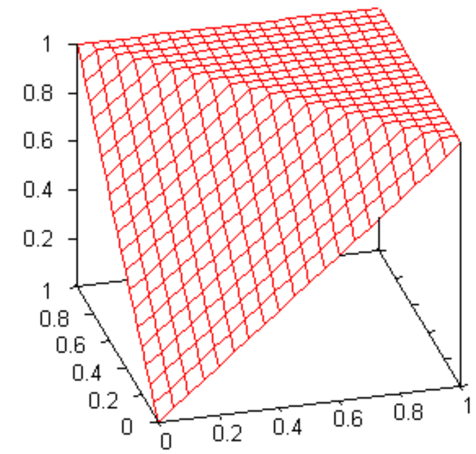
- The interval $[0,1] \subseteq \mathbb{R}$ lies both in C and in P. It represents the sheaf $\hom_{\mathsf{P}}(-,[0,1])$, an object of $\mathscr{P}$.

- A *Riesz MV-algebra* is the unit interval of a unital vector lattice—here, $[0,1]$ is the unit interval of $(\mathbb{R},1)$. Primitive operations:

$$0, \quad x \oplus y \coloneqq (x+y) \wedge 1, \quad \neg x \coloneqq 1-x, \quad r \cdot x \coloneqq rx, \tag{*}$$

  the latter being multiplication by each real scalar $r \in [0,1]$.

- Then $\hom_{\mathsf{P}}(-,[0,1])$ is naturally a Riesz MV-algebra object in $\mathscr{P}$, because the operations (*) are morphisms in the PL site $(\mathsf{P}, \mathsf{J}^*)$. E.g., truncated addition $\oplus\colon [0,1] \times [0,1] \longrightarrow [0,1]$ is the PL map:



What is the theory classified by the PL topos?

- **Baker-Beynon Duality**, 1977. The category of finitely presented unital vector lattices is dually equivalent to P.
- **Chang-Mundici Equivalence** (adapted), 1986. The category of unital vector lattices is equivalent to the category of Riesz MV-algebras
- **Classifier of algebraic theories**, folklore. The presheaf topos $\widehat{\mathsf{P}}$ classifies the theory of Riesz MV-algebras.
- **Extension to the theory of linear Riesz MV-algebras.** Adding the axiom $\forall x \forall y\,((x \leqslant y) \vee (y \leqslant x))$ amounts to asserting: *The square is covered by the triangles above and below the diagonal.*

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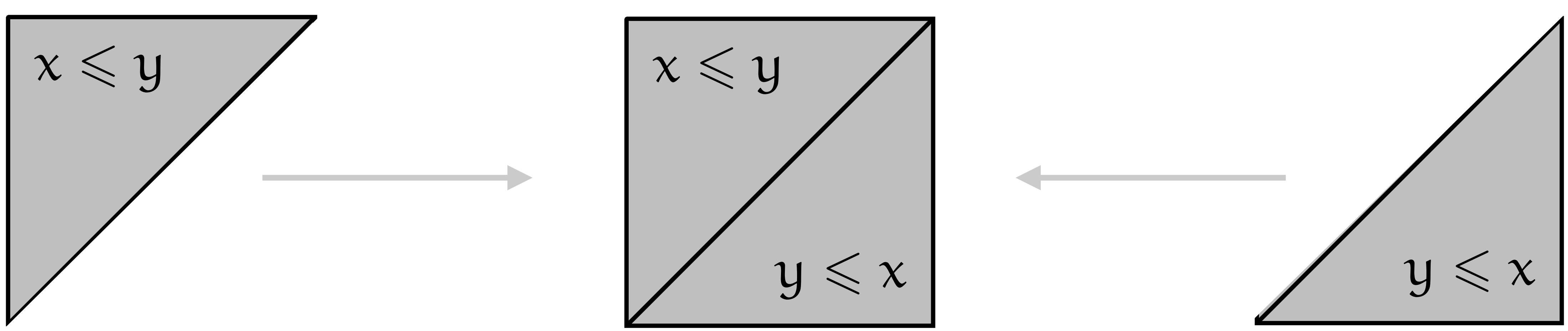

$\mathscr{P}$ classifies the theory of linear Riesz MV-algebras

Theorem. *The PL topos classifies the theory of non-trivial* ($0 \neq 1$) *linearly ordered Riesz MV-algebras.*

Compare (Joyal, when?): *The topos of simplicial sets classifies the theory of nontrivial linear orders with endpoints.* Simplicial sets have a very natural geometric realisation in the PL topos—this is better than realising in Top, cf. Johnstone's topological topos.

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One open question:
What is the intermediate logic of the PL topos?

To tackle this we need to understand Heyting algebras of subobjects in the PL topos. This leads to an important connection with spectra of unital vector lattices, or Riesz MV-algebras.

The Heyting algebras of subobjects in $\mathscr{P}$

Any polyhedron P comes with its distributive lattice $\mathrm{Sub}\,\mathsf{P}$ of subpolyhedra (in fact, a co-Heyting algebra). In the PL topos, $\hom_{\mathsf{P}}(-, \mathsf{P})$ may acquire very many new subobjects.

General theory says that the subobjects of $\hom_{\mathsf{P}}(-, \mathsf{P})$ are in natural bijection with the *closed sieves* on P in the PL site $(\mathsf{P}, \mathsf{J}^*)$.

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Lemma. (i) *For* P *any polyhedron, closed sieves on* P *in the PL site* $(\mathsf{P}, \mathsf{J}^*)$ *are in natural bijection with the ideals of the lattice* $\mathrm{Sub}\,\mathsf{P}$. (ii) *This lattice, in turn, is anti-isomorphic to the lattice of congruences of the Riesz MV-algebra* $\hom_{\mathsf{P}}(\mathsf{P}, [0, 1])$.

The lemma provides a connection between subobjects of representable in the PL topos, and the standard representation theory of unital vector lattices, or Riesz MV-algebras.

The Heyting algebras of subobjects in $\mathscr{P}$

Theorem. *Let*

$$\mathsf{F} \coloneqq \hom_{\mathsf{P}}(-, \mathsf{P})$$

be a representable sheaf in $\mathscr{P}$*, and let*

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be its Baker-Beynon dual Riesz MV-algebra of PL maps $\mathsf{P} \to [0,1]$.

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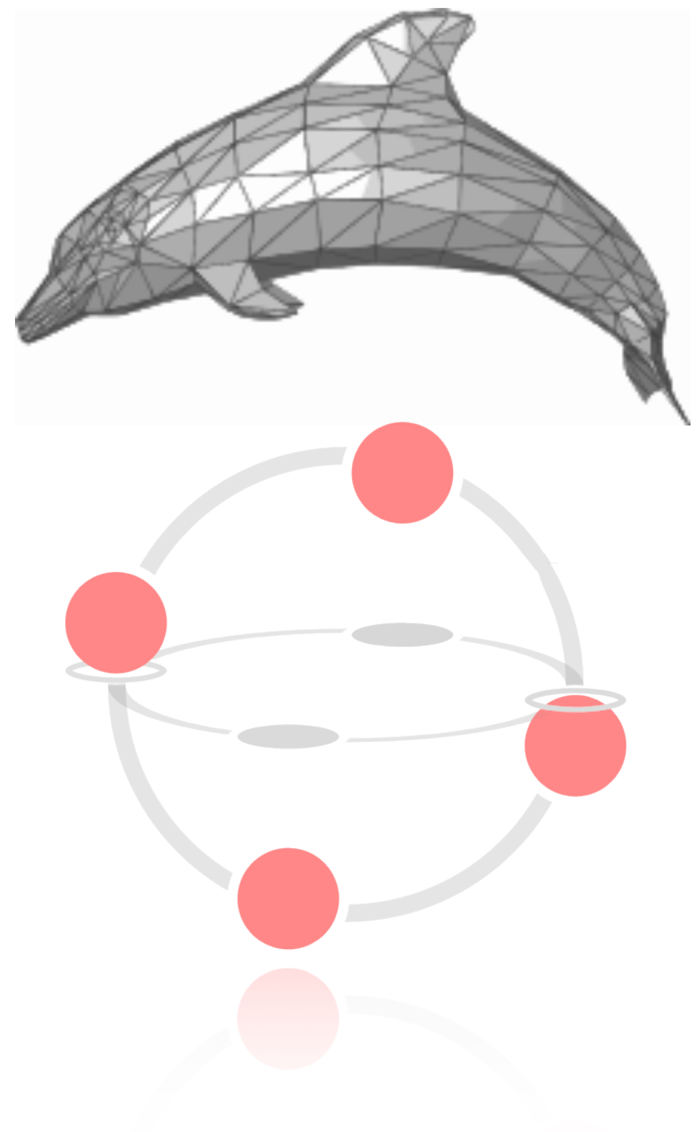
be its Baker-Beynon dual Riesz MV-algebra of PL maps $\mathsf{P} \to [0,1]$. *Then the natural map that sends a closed sieve in* $(\mathsf{P}, \mathsf{J}^*)$ *to the corresponding congruence of* A *induces an isomorphism*

$$\operatorname{Sub}\mathsf{F} \xrightarrow{\cong} \mathscr{O}\left(\operatorname{Spec}\mathsf{A}\right)$$

of the Heyting algebra $\operatorname{Sub}\mathsf{F}$ *with the locale of open sets of the spectral space* $\operatorname{Spec}\mathsf{A}$ *equipped with the dual Stone topology.*



Thank you for your attention.