Two Approaches to Substructural Modal Logic: Some Elementary Observations

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Additional motivations:

- Many-valued models are simpler, but Routley-Meyer models have a clearer epistemic interpretation (support by pieces of information...)
- Many-valued PDL makes very good sense (variables of a non-Boolean type), but the Routley-Meyer modelling is somewhat more tangible...

Results so far:

- Turning countermodels to φ of one kind to countermodels of the other kind.
- A class of lattice-based Kripke frames giving the logic of all Routley-Meyer frames.

FL-type: $\wedge, \vee, \backslash, \bullet, /, 1, 0$;   mFL-type adds unary $\Box$.

A modal FL-algebra is a mFL-type algebra $\boldsymbol{M}$ where the FL-type reduct is a FL-algebra

- $\langle A, \wedge, \vee \rangle$ lattice
- $\langle A, \bullet, 1 \rangle$ monoid
- $a \bullet b \leq c$ iff $b \leq a \backslash c$ iff $a \leq c/b$

and

- $\Box(a \wedge b) = \Box a \wedge \Box b$

Formula algebras: $\boldsymbol{Fm}$ is an absolutely free mFL-type algebra with a countable set Prop of generators; $\boldsymbol{F}$ is an absolutely free FL-type algebra over Prop.

A Routley-Meyer frame is $\mathscr{F} = \langle S, \leq, T, F, R_3, R_2 \rangle$

- $(S, \leq)$ poset
- T, F subsets of S upwards closed under $\leq$
- R_3 ternary, antitone in first two positions, monotone in third
- R_2, binary, antitone in first position, monotone in second
- $s \leq t$ iff $\exists u \in T : R_3 sut$
- $R_3 stuw$ iff $R_3 s(tu)w$

A Routley-Meyer frame is $\mathscr{F} = \langle S, \leq, T, F, R_3, R_2 \rangle$

The full complex algebra of $\mathscr{F}$ is

$\mathscr{F}^{\mathfrak{ca}} = \langle Up(\mathscr{F}), \cap, \cup, \backslash_{\mathfrak{ca}}, \bullet_{\mathfrak{ca}}, /_{\mathfrak{ca}}, \Box_{\mathfrak{ca}}, 1_{\mathfrak{ca}}, 0_{\mathfrak{ca}} \rangle$

- $Up(\mathscr{F})$ upwards closed subsets of $\mathscr{F}$
- $X \backslash_{\mathfrak{ca}} Y = \{s \,;\, \forall t, u : R_3tsu \;\;\&\;\; t \in X \implies u \in Y\}$
- $Y /_{\mathfrak{ca}} X = \{s \,;\, \forall t, u : R_3stu \;\;\&\;\; t \in X \implies u \in Y\}$
- $X \bullet_{\mathfrak{ca}} Y = \{s \,;\, \exists t, u : R_3tus \;\;\&\;\; t \in X \;\;\&\;\; u \in Y\}$
- $\Box_{\mathfrak{ca}} X = \{s \,;\, \forall t : R_2st \implies t \in X\}$
- $1_{\mathfrak{ca}} = T$ and $0_{\mathfrak{ca}} = F$

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A model based on $\mathscr{F}$ is $\mathscr{M} = \langle \mathscr{F}, V \rangle$ where V : Prop $\longrightarrow Up(\mathscr{F})$; the latter extends to a hom. $\bar{V}$ from $\boldsymbol{Fm}$ to $\mathscr{F}^{\mathfrak{ca}}$. Validity as $T \subseteq \bar{V}(\varphi)$ for all V.

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Fact. $\mathscr{F}^{\mathfrak{ca}}$ a modal FL algebra; φ valid in $\mathscr{F}$ iff valid in $\mathscr{F}^{\mathfrak{ca}}$.

The Routley-Meyer frame of $\boldsymbol{M}$ is

$$\boldsymbol{M}_{\mathfrak{rm}} = \langle Pr(\boldsymbol{M}), \subseteq, T_{\mathfrak{rm}}, F_{\mathfrak{rm}}, R^3_{\mathfrak{rm}}, R^2_{\mathfrak{rm}} \rangle$$

- $Pr(\boldsymbol{M})$ set of prime filters on $\boldsymbol{M}$
- $T_{\mathfrak{rm}} = \{P \,;\, 1 \in P\}$
- $F_{\mathfrak{rm}} = \{P \,;\, 0 \in P\}$
- $R^3_{\mathfrak{rm}} = \{\langle P, P', Q\rangle \,;\, (\forall a, b \in \boldsymbol{M} : a \in P \;\;\&\;\; b \in P' \implies a \bullet b \in Q)\}$
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Theorem 1.

(a) $h : a \mapsto \{P \,;\, a \in P\}$ embeds $\boldsymbol{M}$ into $(\boldsymbol{M}_{\mathfrak{rm}})^{\mathfrak{ca}}$.

(b) φ valid in $\boldsymbol{M}$ if valid in $(\boldsymbol{M}_{\mathfrak{rm}})^{\mathfrak{ca}}$.

(c) φ valid in $\boldsymbol{M}$ if valid in $\boldsymbol{M}_{\mathfrak{rm}}$.

Let $\boldsymbol{A}$ be a complete FL-algebra. An $\boldsymbol{A}$-based frame is $\mathcal{F}_{\boldsymbol{A}} = \langle \langle S, R \rangle, \boldsymbol{A} \rangle$, where $\langle S, R \rangle$ is a Kripke frame.

A model based on $\mathcal{F}_{\boldsymbol{A}}$ is $\mathcal{M}_{\boldsymbol{A}} = \langle \mathcal{F}_{\boldsymbol{A}}, v \rangle$ where $v : \mathrm{Prop} \to (S \to \boldsymbol{A})$.

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We define $\bar{v} : \boldsymbol{Fm} \to (S \to \boldsymbol{A})$:

- $\bar{v}_\varphi$ an FL-homomorphism
- $\bar{v}_{\Box\varphi}(s) := \bigwedge\{\bar{v}_\varphi(t) \; ; \; Rst\}$

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The full complex algebra of $\mathcal{F}_{\boldsymbol{A}}$ is $\mathcal{F}_{\boldsymbol{A}}^{\mathfrak{ca}} = \langle \boldsymbol{A}^S, \{\nabla^{\mathfrak{ca}} \; ; \; \nabla \in \text{ mFL operators}\} \rangle$ where

- $\big(\nabla^{\mathfrak{ca}}(f_1, \ldots, f_n)\big)(s) = \nabla^{\boldsymbol{A}}(f_1(s), \ldots, f_n(s))$ if ∇ is FL op.
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Fact. $\mathcal{F}^{\mathfrak{ca}}_{\boldsymbol{A}}$ is a mFL-algebra; φ valid in $\mathcal{F}_{\boldsymbol{A}}$ iff valid in $\mathcal{F}^{\mathfrak{ca}}_{\boldsymbol{A}}$.

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Theorem 2. φ valid in $\mathcal{F}_{\boldsymbol{A}}$ if valid in $(\mathcal{F}^{\mathfrak{ca}}_{\boldsymbol{A}})_{\mathfrak{rm}}$.

The lattice-based frame of $\boldsymbol{M}$ with non-modal reduct $\boldsymbol{A}$ is
$\boldsymbol{M}_{\mathfrak{lb}} = \langle Hom(\boldsymbol{A}^{\boldsymbol{M}}), R, \boldsymbol{A} \rangle$

- Rhg iff $\forall a \in \boldsymbol{M} : h(\Box a) = g(a)$ (*not* $h(\Box a) \leq g(a)$!)

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Theorem 3.

(a) $\theta : a \mapsto f_a$, where $f_a(h) = h(a)$, embeds $\boldsymbol{M}$ into $(\boldsymbol{M}_{\mathfrak{lb}})^{\mathfrak{ca}}$.
(b) φ valid in $\boldsymbol{M}$ if valid in $(\boldsymbol{M}_{\mathfrak{lb}})^{\mathfrak{ca}}$.
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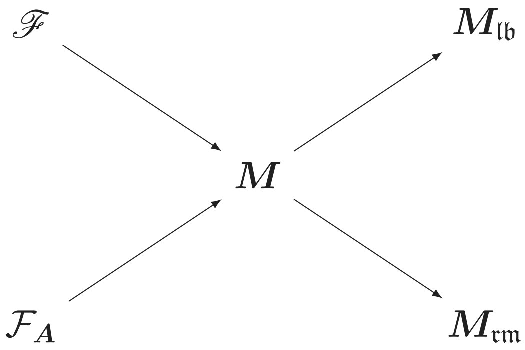
Theorem 4. φ valid in $\mathscr{F}$ if valid in $(\mathscr{F}^{\mathfrak{ca}})_{\mathfrak{lb}}$.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If $\boldsymbol{A}$ is complete distributive, then $(\mathcal{F}^{\mathfrak{ca}}_{\boldsymbol{A}})_{\mathfrak{rm}}$ is a Routley-Meyer frame.

Theorem 5. The logic of all Routley-Meyer frames is the logic of all Kripke frames based on complete distributive FL-algebras.

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In general, if

$$\mathscr{F} \in \mathscr{K} \implies \mathscr{F}^{\mathfrak{ca}-} \in \mathsf{K}$$
$$\boldsymbol{A} \in \mathsf{K} \implies (\mathcal{F}^{\mathfrak{ca}}_{\boldsymbol{A}})_{\mathfrak{rm}} \in \mathscr{K}$$

then the logic of $\mathscr{K}$ is the logic of Kripke frames based on K.