A temporal interpretation of intuitionistic quantifiers

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Nice, TACL 2019

June 21, 2019

Gödel translation

Propositional version (from IPC to S4)

$$\begin{aligned}
\bot^t &= \bot \\
p^t &= \Box p && \text{for each propositional letter } p \\
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Theorem (McKinsey and Tarski 1948)

$\mathbf{IPC} \vdash \varphi$ *iff* $\mathbf{S4} \vdash \varphi^t$

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Predicate version (from IQC to QS4):

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(\forall x \varphi)^t = &\quad \Box \forall x \varphi^t \\
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Theorem (Rasiowa and Sikorski 1963)

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Later, in 1968, Schütte gave a proof using Kripke semantic.

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- $\exists p \to \forall\exists p$;
- $\exists\forall p \to \forall p$.

Theorem (Bull 1966)

MIPC *axiomatizes the monadic fragment of* **IQC**.

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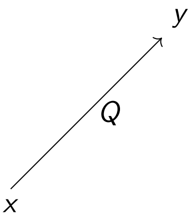

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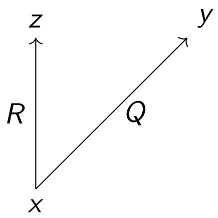

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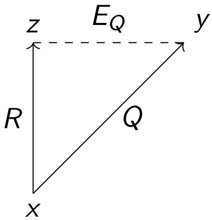

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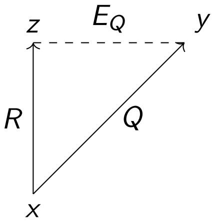

where zE_Qy means zQy and yQz.

Relational semantic for **MIPC** through Ono frames

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Theorem (Ono 1977, Bezhanishvili 1998)

MIPC *is complete with respect to the class of Ono frames.*

MS4

Definition (Fischer-Servi 1977)

MS4 is obtained from the fusion **S4** $\otimes$ **S5** by adding the left commutativity axiom

$$\Box\forall p \to \forall\Box p$$

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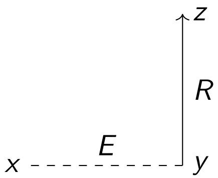

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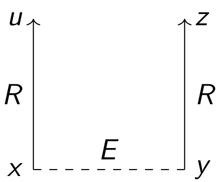

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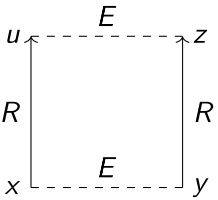

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Theorem (Esakia)

MS4 *is complete with respect to the class of* **MS4**-*frames.*

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Given an Ono frame (X, R, Q), we obtain the **MS4**-frame (X, R, E_Q).

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Vice versa from an **MS4**-frame (X, R, E) we obtain the Ono frame $(X, R, R \circ E)$.

This correspondence is not a bijection. Indeed, $Q = R \circ E_Q$ but $E \neq E_{R \circ E}$ in general. But it restricts to a bijection on canonical frames.

Monadic version of the Gödel translation

The Gödel translation of **IQC** into **QS4** restricts to a translation of **MIPC** into **MS4**.

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Main idea

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We do so by defining the temporal logic **TS4** and adjust the Gödel translation in such a way it remains full and faithful when we translate **MIPC** into **TS4**.

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Definition (Esakia 1978, Wolter 1998)

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Where $\blacklozenge_F = \neg\blacksquare_F\neg$ and $\blacklozenge_P = \neg\blacksquare_P\neg$.

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- $\blacksquare_F q \to \Box \blacksquare_F q$;
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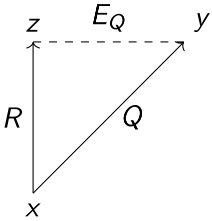

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Theorem

TS4 *is complete with respect to the class of* **TS4**-*frames.*

Translation of **MIPC** into **TS4**

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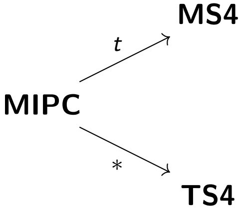
Theorem

MIPC $\vdash \varphi$ *iff* **TS4** $\vdash \varphi^*$

The diagram of translations



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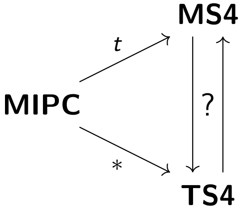

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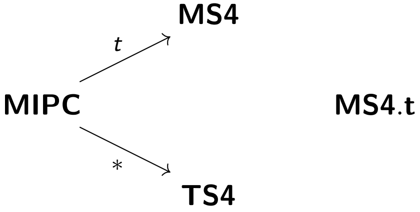

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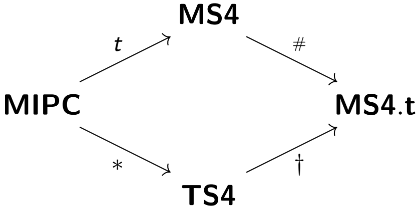

MS4.t

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MS4.**t** is obtained from the fusion **S4**.**t** $\otimes$ **S5** by adding the left commutativity axiom

$$\Box_F \forall p \to \forall \Box_F p.$$

Where the two temporal **S4**-operators are denoted by $\Box_F$ and $\Box_P$ and the **S5**-operator by $\forall$.

Translation of **MS4** into **MS4.t**

We can think of **MS4.t** as an extension of **MS4**.

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Translation of **TS4** into **MS4.t**

$$(\Box\varphi)^\dagger = \Box_F \varphi^\dagger$$
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$$(\blacksquare_P \varphi)^\dagger = \forall \Box_P \varphi^\dagger$$

Theorem

TS4 $\vdash \varphi$ *iff* **MS4.t** $\vdash \varphi^\dagger$

Finite model property for **MS4.t**

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MS4.t *has the finite model property.*

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Thus we obtain an uniform approach for the proof of the finite model property for all these logics.

PREDICATE CASE

Main idea

$$(\forall x \varphi)^t = \Box_F \forall x \varphi^t$$
$$(\exists x \varphi)^t = \Diamond_P \exists x \varphi^t$$

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To make these ideas work we use Corsi's semantic (2002).

Thanks for your attention!

$\mathbf{Q^{\circ}S4.t + CBF_F}$

Let us consider a predicate language containing two modal operators $\Box_F$ and $\Box_P$. The logic $\mathbf{Q^{\circ}S4.t + CBF_F}$ is the one defined by the axioms

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \qquad \forall y(\forall x\varphi(x) \to \varphi(y/x))$$

$$\forall x\forall y\varphi \leftrightarrow \forall y\forall x\varphi \qquad \forall x(\varphi \to \psi) \to (\forall x\varphi \to \forall x\psi)$$

$$\exists\top \qquad \varphi \to \forall x\varphi \quad \text{provided } x \text{ not free in } \varphi.$$

That is closed under necessitation, universal generalization and modus ponens.

Semantic for $\mathbf{Q}^\circ\mathbf{S4.t} + \mathbf{CBF_F}$

We consider Kripke frames $(X, R, \{D_w\}, U)$ with:

- a **S4**-Kripke frame (X, R);
- a set U, called outer domain;
- a map associating to any world $w \in X$ a set $D_w \subseteq U$ called inner domain at w;
- wRv implies $D_w \subseteq D_v$.

Theorem (Corsi 2002)

$\mathbf{Q}^\circ\mathbf{S4.t} + \mathbf{CBF_F}$ *is complete with respect to the class of frames described above where the quantified variables are interpreted in the inner domains* D_w*'s while the free variables and constants are interpreted inside the outer domain* U.

Translation of **IQC** into $\mathbf{Q^{\circ}S4.t} + \mathbf{CBF_F}$

The following is a full and faithful translation of **IQC** into $\mathbf{Q^{\circ}S4.t} + \mathbf{CBF_F}$.

$$
\begin{aligned}
P^t &= \Box_F P \\
(\varphi \to \psi)^t &= \Box_F(\neg\varphi^t \vee \psi^t) \\
(\forall x\varphi)^t &= \Box_F \forall x \varphi^t \\
(\exists x\varphi)^t &= \Diamond_P \exists x \varphi^t
\end{aligned}
$$

This translation is full and faithful only when restricted to sentences. This is because only the universal closure of the universal instantiation axiom is provable in $\mathbf{Q^{\circ}S4.t} + \mathbf{CBF_F}$. We cannot have completeness with respect to the class of frames described above if we include the universal instantiation axiom.

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$$\begin{aligned} P^t &= \Box_F P \\ (\varphi \to \psi)^t &= \Box_F(\neg\varphi^t \vee \psi^t) \\ (\forall x\varphi)^t &= \Box_F \forall x \varphi^t \\ (\exists x\varphi)^t &= \exists x \Diamond_P \varphi^t \end{aligned}$$

This translation is full and faithful only when restricted to sentences. This is because only the universal closure of the universal instantiation axiom is provable in $\mathbf{Q}^{\circ}\mathbf{S4.t} + \mathbf{CBF_F}$. We cannot have completeness with respect to the class of frames described above if we include the universal instantiation axiom.

Constant domains

The logic **QS4.t** is obtained by adding the universal instantiation axiom to $\mathbf{Q^{\circ}S4.t} + \mathbf{CBF_F}$.

It is complete with respect to the class of frames with inner domains coinciding with the outer domain and therefore constant.

The previous translation gives a full and faithful translation of $\mathbf{IQC} + (\forall x\,(\forall x\,\varphi \lor \psi)) \to (\forall x\,\varphi \lor \forall x\,\psi)$ into **QS4.t**. In this case the translation works for all the sentences.