Duality - Introduction

- human cognition: verbal vs visual
- in mathematics: algebra vs geometry
- in logic / comp sc: syntax vs semantics
- Duality: link between these perspectives

  ↳ can/will be made very precise using category theory

- Establishing a duality / dual equivalence should be a starting point!

- Course:
  1. Stone Duality for BAs
  2. Priestley Duality
  3. Modal Duality

- Note: very introductory!

Material

- Mai Gehrke & Sam van Gool:
  Duality Theory
  → tinyurl.com/dualitybook

- Yde Venema
  Algebra and Coalgebra
  Handbook of Modal Logic
  → web page

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I Stone Duality for Boolean algebras.

0. Introduction: BA abstract from power sets.

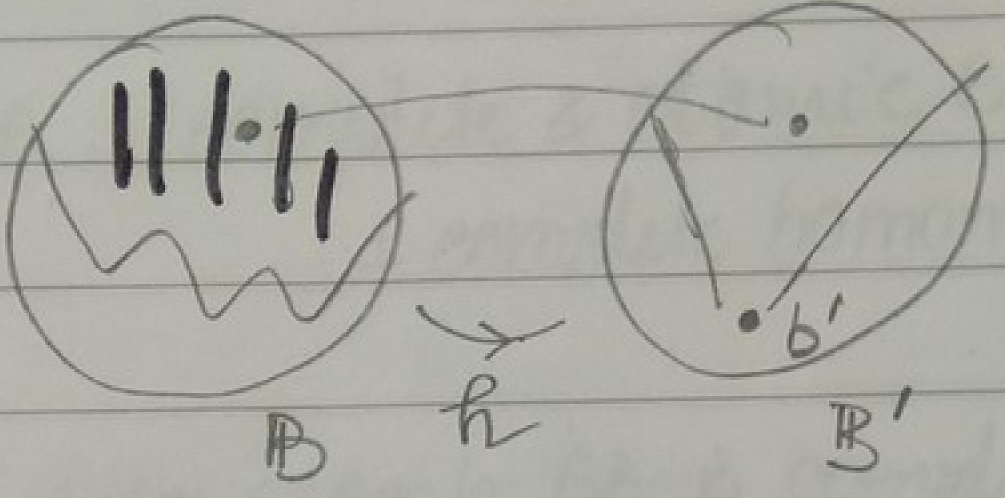
- Given X, def $\mathbb{P}X := (PX, \emptyset, \sim_X, \cup) =: X^+$
  Given $f: X \to Y$, $Pf: PX \to PY$ direct image
  $f^+ := \check{\mathbb{P}}f: PY \to PX$ inverse image
  Prop: $(\cdot)^+ : \mathrm{Set} \xrightarrow{op} \mathbb{B}A$.

- Reverse? singletons

1. Finite case: finite BAs are atomic.

- Def: atom, $\mathbb{B}_+$
- Prop: $X \cong (\mathbb{P}(X))_+$
- Prop (properties of atoms)
  - a atom $\Leftrightarrow$ $a \leq \bigvee C$ iff $a \leq c$, some $c \in C$
  - $b = \bigvee\{a \in \mathbb{B}_+ \mid a \leq b\}$
  - $b \neq \emptyset \Rightarrow \exists a \in \mathbb{B}_+.\ a \leq b$.
- Prop: $\mathbb{B} \cong (\mathbb{B}_+)^+$. REPRESENTATION THM
- Def: $\hat{\cdot} : \mathbb{B} \to (\mathbb{B}_+)^+$
  $\hat{b} := \{a \in \mathbb{B}_+ \mid a \leq b\}$

- **morphisms**: given $h : \mathbb{B} \to \mathbb{B}'$
  think of $(\mathbb{B}_+)^+ \to (\mathbb{B}'_+)^+$



Is there a $h_+ : \mathbb{B}'_+ \to \mathbb{B}_+$ s.t. ...

- $h_+(b) := \bigwedge \{ b \mid b' \leq hb \}$
- **Prop**
  - $b' \leq h(h_+(b))$
  - $b' \leq hb$ iff $h_+(b') \leq b$ ADJUNCTION
  - h_+ preserves joins!

- **Prop** $h_+ : \mathbb{B}'_+ \to \mathbb{B}_+$
  If $h_+(a') \leq \bigvee C \Rightarrow a' \leq h(\bigvee C) = \bigvee h[C]$
  $\Rightarrow a' \leq h(c)$ some $c \in C$
  $\Rightarrow h_+(a') \leq c$ some $c \in C$.

Thm The functors $(\cdot)^+$ and $(\cdot)_+$ constitute a dual equivalence between the categories FinSet and FinBA.

$$\begin{tikzcd} \mathrm{FinBA} \arrow[r, bend right, "(\cdot)_+"'] & \mathrm{FinSet} \arrow[l, bend right, "(\cdot)^+"'] \end{tikzcd}$$

- Note: reverse arrows!

2. Discrete case :

$$\begin{tikzcd} ? \arrow[r, bend right, "?"'] & \mathrm{Set} \arrow[l, bend right, "(\cdot)^+"'] \end{tikzcd}$$

▷ CABA : complete & atomic BA's
+ complete homomorphisms.

▷ Not every atomic BA is complete
—— complete BA is atomic
Not every BA hom between CABA's is complete
(eg use uf).

two ideas!

3. Topological Duality : BA $\rightleftarrows$?

▶ Arbitrary BAs may not have atoms → ~~create~~ simulate
atom a of $\mathbb{B}$ ~ $\{b \in B \mid a \leq b\}$.

▶ Def : filter/ideal
ultrafilter $\mathbb{B}_{\circ}$

▶ Thm [STONE REP] $\mathbb{B} \hookrightarrow (\mathbb{B}_{\circ})^{+}$ by
$\varphi : a \mapsto \hat{a} := \{u \in \mathbb{B}_{\circ} \mid a \in u\}$

Pf - homomorphism : prop of (ultra) filters
- injective : if $a \neq b$ WLOG $a \not\leq b \overset{?}{\Rightarrow} \exists u (a \notin u, b \in u)$

Ultrafilter Theorem.
$a \notin F \in Fil(\mathbb{B}) \Rightarrow \exists u \supseteq F, a \notin u$.

▶ How to retrieve $\mathbb{B}$ from $(\mathbb{B}_{\circ})^{+}$? Topologize!
generate topology from $Im(\varphi)$

▶ Def A Boolean space is a comp Hd space with clopen basis

BS := Boolean spaces + continuous maps

Prop $f : \mathbb{X} \to \mathbb{X}'$ ct, $\mathbb{X}, \mathbb{X}'$ Boolean Then $f^{-1} : Clp(\mathbb{X}') \to Clp(\mathbb{X})$

▸ **Prop** $(\cdot)^*$ is a functor $(\cdot)^* : BS \to BA$, contravariant
with $\mathbb{X}^* := (\mathrm{Clp}\,\mathbb{X}, \emptyset, \sim_X, \cup)$
$f^* := f^{-1}$

▸ **Prop** $(\cdot)_*$ is a ctr functor $(\cdot)_* : BA \to BS$
with $\mathbb{B}_* := (\mathbb{B}_\circ, \tau_{\mathbb{B}})$
Pf $h_* : u' \mapsto \{a \in A \mid ha \in u'\}$
- well-defined: $\mathbb{B}_*$ is a Booleanspace
: $h_* : \mathbb{B}'_\circ \to \mathbb{B}_\circ$
h_* is continuous.

▸ **Prop** $\mathbb{B} \cong (\mathbb{B}_*)^*$ via $a \mapsto \hat{a}$
Pf • hom/inj: as in Stone Rep
• surj: Need: $\mathrm{Clp}(\mathbb{B}_*) = \mathrm{Im}(e)$.

▸ **Prop** $\mathbb{X} \cong (\mathbb{X}^*)_*$ via $x \mapsto \{C \in \mathrm{Clp}(\mathbb{X}) \mid x \in C\}$
P

▸ **Theorem (Stone duality)**: $BA \underset{(\cdot)^*}{\overset{(\cdot)_*}{\rightleftarrows}} BS$

Prop $h: \mathbb{B} \to \mathbb{B}'$ hom $\Rightarrow h_*: \mathbb{B}'_* \to \mathbb{B}_*$ cont

Pf Take $C \in \mathrm{Clp}(\mathbb{B}_*)$ say $C = \hat{b}$.

$$\begin{aligned} h_*^{-1}(C) &= \{u' \in \mathbb{B}'_\circ \mid h_*(u') \in \hat{b}\} \\ &= \{u' \in \mathbb{B}'_\circ \mid b \in h_*(u')\} \\ &= \{u' \in \mathbb{B}'_\circ \mid h(b) \in u'\} \\ &= \widehat{h(b)} \qquad\qquad \in \mathrm{Clp}(\mathbb{B}'_*). \end{aligned}$$

4 Comments / Remarks

- Duality: • reverse arrows
  - complementary intuitions if both cat's are set-based (concrete)

- Stone duality is a natural duality
- May obtain top duality from finite duality.