II Variations / Generalizations : DL

A) Distributive Lattices $\mathbb{D} = (D, \bot, \top, \vee, \wedge)$.

1) Birkhoff duality.

- PO : category of partial orders & order preserving maps.
- $D \subseteq P$ is a downset of the PO $\mathbb{P} = (P, \leq)$
  if $q \leq p \in D \Rightarrow q \in D$.
  - $\mathcal{D}(\mathbb{P})$: downsets of $\mathbb{P}$.

    $\mathbb{P}^+ := (\mathcal{D}(\mathbb{P}), \emptyset, P, \cup, \cap)$.

    with $f : \mathbb{P} \to \mathbb{P}'$, define $f^+ := f^{-1}$.

Prop. $(\cdot)^+$ is a functor $(\cdot)^+ : PO \to DL$.

- Back from Fin DL to PO : need "atoms".

  $j \in D$ is a join irreducible of finite $\mathbb{D}$ if

  $j = \bigvee S \Rightarrow j \in S$.

  $j \in D$ is a join prime of a finite DL $\mathbb{D}$ if

  $j \leq \bigvee S \Rightarrow j \leq s$, some $s \in S$.

Prop In distributive lattice: j.i $\Leftrightarrow$ j.p.

▷ $\mathbb{D}_+ := (\mathcal{J}(\mathbb{D}), \leq)$.

▷ <u>Prop</u>: $\mathbb{D} \cong (\mathbb{D}_+)^+$ via $e: d \to \hat{d} := \{j \in \mathcal{J}(\mathbb{D}) \mid j \leq d\}$.

Pf. • e homomorphism: ✓

▷ • e injective: need $a \not\leq b \Rightarrow \exists j \in \mathcal{J}(\mathbb{D}): j \leq a, j \not\leq b$.

• e surjective: with $A \in \mathcal{D}(\mathcal{J}(\mathbb{D}))$, show $\widehat{\bigvee A} = A$.

▷ <u>Prop</u>: $\mathbb{P} \cong (\mathbb{P}^+)_+$, via $p \mapsto \downarrow p$.

Pf: Need to show: $\downarrow p \in \mathcal{J}(\mathbb{P}^+)$. & v.v.

▷ <u>Morphisms</u>: given $h: \mathbb{D} \to \mathbb{D}'$, need $h_+: \mathbb{D}'_+ \to \mathbb{D}_+$

Same as for BA - case!

$h_+(d') := \bigwedge \{d \in D \mid d' \leq hd\}$.

▷ <u>Thm</u> $\quad \mathrm{FinDL} \underset{(\cdot)^+}{\overset{(\cdot)_+}{\rightleftarrows}} \mathrm{FinPO}$ is a duality.

2) Priestley Duality $\quad DL \equiv^{op}$?

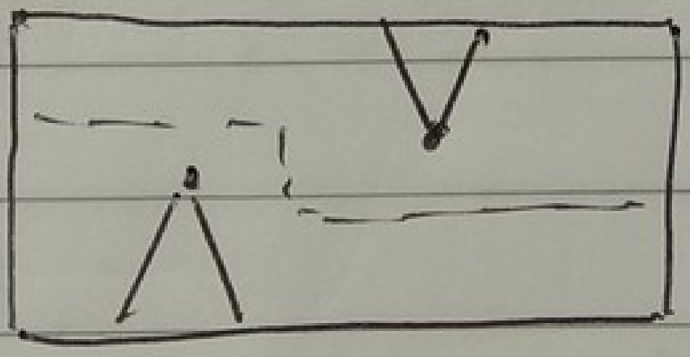
- Ordered space: $(X, \mathcal{T}, \leq)$ with $\leq \subseteq X \times X$ closed.

Compact ordered space: COS

↳ arrows: cont. ord. pres. maps

- Basic facts $\mathbb{X} = (X, \mathcal{T}, \leq)$ a COS
  - $(X, \mathcal{T})$ is Hausdorff
  - F closed $\Rightarrow$ $\downarrow F, \uparrow F$ closed; e.g.: $\downarrow x, \uparrow x$
  - order separation:

- Def: Priestley space COS $\mathbb{P} = (P, \mathcal{T}, \leq)$ which is totally order disconnected:



- From PS to DL: Down

$$\mathbb{P}^* := (\mathrm{Cl}\cancel{\mathrm{Up}}(\mathbb{P}), \emptyset, P, \cup, \cap)$$

$$f^* := f^{-1}.$$

$\sim\hat{a} = \mathrm{PrFil}(D) \setminus \hat{a}$.

▷ From DL to PS:

$$\mathbb{D}_* := (\mathrm{PrFil}(\mathbb{D}), \pi_{\mathbb{D}}, \supseteq)$$

- prime filters: $a \vee b \in F \Rightarrow a \in F$ or $b \in F$
- $\pi_{\mathbb{D}}$: generate from subbasis $\{\hat{a}, \sim\hat{a} \mid a \in D\}$.

▷ Prop If $\mathbb{D} \in DL$ then $\mathbb{D}_* \in PS$.

Pf key observation: compactness

▷ Thm $DL \underset{(\cdot)^*}{\overset{(\cdot)_*}{\rightleftarrows}} PS$ is a cat. duality.

3) Comments

- Stone duality for BA: special case.
- Stone duality for DL
- Heyting algebras

B) Compact Hausdorff spaces : KHaus

$$? \rightleftarrows \mathrm{KHaus}$$

Frames $\mathbb{F} = (F, \wedge, \bigvee)$

$(F, \wedge, \vee, \bigsqcup)$.

compact regular.