III Modal Duality

1) Key categories

- **Kripke frames** (semantics)
  - $\mathfrak{S} = (S, R)$ with $R \subseteq S \times S$, or: $R : S \to PS$
  - bounded morphisms:

$$\begin{tikzcd} S \arrow[r, "R"'] \arrow[d, "f"'] & \mathcal{P}(S) \arrow[d, "\mathcal{P}f"] \\ S' \arrow[r, "R'"'] & \mathcal{P}(S') \end{tikzcd}$$

i.e. (forth) $Rs_0s_1 \to Rfs_0fs_1$

(back) $R'fs_0s_1' \to \exists s_1 (Rs_0s_1 \,\&\, fs_1 = s_1')$.

- **Modal algebras** (syntax, ...
  - $\mathbb{A} = (A, \bot, -, \vee, \Diamond)$ with $(A, \bot, -, \vee) \in BA$ & $\Diamond : A \to A$ preserving finite joins.
  - arrows: obvious homomorphisms.

2) From Kripke structures to modal algebras

- $\mathfrak{S}^+ := (\mathbb{P}(S), \langle R \rangle)$ where $\langle R \rangle X := \{ y \in S \mid Ryx, \text{some } x \in X \}$.
- $f^+ := f^{-1}$

semantics of $\Diamond$!

Prop. $(\cdot)^+$ is a functor $(\cdot)^+ : KS \to MA$.

3) Finite $^{\text{modal}}$ Duality.

- Given finite MA $\mathbb{A}$, put $R: \mathbb{A}_+^* \to \wp(A_+)$

$$R_\Diamond(a') := \{a \mid a' \leq \cancel{f(a)} \cdot \Diamond a\}$$

~~$\mathbb{A}_+$~~

$$(\mathbb{B}, \Diamond)_+ := (\mathbb{B}_+, R_\Diamond).$$

- Prop: if $h: \mathbb{A} \to \mathbb{A}'$ then $h_+ : \mathbb{A}'_+ \to \mathbb{A}_+$.

Pf

$$\begin{tikzcd}
A'_+ \arrow[r, "h_+"] \arrow[d, "R_\Diamond'"'] & A_+ \arrow[d, "R_\Diamond"] \\
P(A'_+) \arrow[r, "P(h_+)"'] & P(A_+)
\end{tikzcd}.$$

- Thm: $\text{FinMA} \underset{(\cdot)^+}{\overset{(\cdot)_+}{\rightleftarrows}} \text{FinKS}$ is a duality.

4) Topological Modal Duality

piggyback on Stone!

- For MA $\mathbb{A} = (\mathbb{B}, \Diamond)$ define $R_\Diamond \subseteq \mathbb{B}_\circ \times \mathbb{B}_\circ$

  $R_\Diamond uv :\Leftrightarrow$ for all a: $a \in v \Rightarrow \Diamond a \in u$.

  For $f : \mathbb{A} \to \mathbb{A}'$

  $f_\circ u' := \{a \in A \mid fa \in u'\}$.

Prop $(\cdot)_\circ$ is a functor $(\cdot)_\circ : \mathrm{MA} \to \mathrm{KF}$.

Pf key observation:

if $f : \mathbb{A} \to \mathbb{A}'$ then $f_\circ : \mathbb{A}'_\circ \to \mathbb{A}_\circ$.

Prop $\overbrace{\mathbb{A} \hookrightarrow (\mathbb{A}_\circ)^*}^{\text{Jó-Ta Rep}}$ via $a \mapsto \hat{a} := \{u \in \mathbb{A}_\circ \mid a \in u\}$.

Pf key observation:

$\widehat{\Diamond a} = \langle R_\Diamond \rangle \hat{a}$.

- Topological Kripke Structures $\mathbb{S} = (S, \mathcal{T}, R)$
  - $(S, \mathcal{T})$ Boolean space
  - $\langle R \rangle : Clp(\mathcal{T}) \to Clp(\mathcal{T})$
  - $R(s)$ is closed, all $s \in S$.

▷ <u>Prop</u> $(\cdot)^*$ is a functor $(\cdot)^* : \mathrm{TKF} \to \mathrm{MA}$

where $\mathfrak{S}^* := (\mathrm{Clp}(\mathfrak{S}), \emptyset, \sim_S, \cup, \langle R \rangle)$

$$f^* := f^{-1}$$

Pf Key observation

$$f^*(\langle R' \rangle U') = \langle R \rangle f^* U'.$$

▷ <u>Prop</u> $(\cdot)_*$ is a functor $\mathrm{MA} \to \mathrm{TKS}$

where $\mathbb{A}_* := (\mathbb{A}_\bullet, \tau_{\mathbb{A}}, R_\diamond).$

$$f_* := f_\ast$$

Pf Key obs:

f_* is a bounded morphism.

▷ <u>Prop</u> $e_{\mathbb{A}} : a \mapsto \hat{a}$ is a natural iso : $\mathbb{A} \mapsto (\mathbb{A}_*)^*$.

Pf: piggyback

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "e_{\mathbb{A}}"] \arrow[d, "f"'] & (\mathbb{A}_*)^* \arrow[d, "(f_*)^*"] \\
\mathbb{A}' \arrow[r, "e_{\mathbb{A}'}"'] & (\mathbb{A}'_*)^*
\end{tikzcd}$$

▷ <u>Prop</u> similar $\mathfrak{S} \cong (\mathfrak{S}^*)_*$.

▷ <u>Theorem</u> $(\cdot)_* : \mathrm{MA} \rightleftarrows \mathrm{TKS} : (\cdot)^*$ is a duality!

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5) Comments

- canonical extensions
- boolean algebras with operators
- ——— " ——— operations
- distributive lattices w — " — .

- Heyting algebras.