A POLYLOGARITHMIC MEASURE ASSOCIATED WITH A PATH ON \( \mathbb{P}^1 \setminus \{0, 1, \infty\} \) AND A \( p \)-ADIC HURWITZ ZETA FUNCTION

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Abstract. With every path on \( \mathbb{P}^1 \setminus \{0, 1, \infty\} \) there is associated a measure on \( \mathbb{Z}_p \). The group \( \mathbb{Z}_p^\times \) acts on measures. We consider two measures. One measure is associated to a path from \( 0 \) to a root of unity \( \xi \) of order prime to \( p \). Another measure is associated to a path from \( 0 \) to \( \xi^{-1} \) and next it is acted by \( -1 \in \mathbb{Z}_p^\times \). We show that the sum of these measures can be defined in a very elementary way. Integrating against this sum of measures we get \( p \)-adic Hurwitz zeta functions constructed previously by Shiratani.

0. Introduction

Let \( K \) be a number field, let \( z \in \mathbb{P}^1(K) \setminus \{0, 1, \infty\} \) and let \( \gamma \) be a path on \( \mathbb{P}^1_K \setminus \{0, 1, \infty\} \) from \( 0 \) to \( z \), i.e. an isomorphism of the corresponding fiber functors. Let \( p \) be a fixed prime number. The Galois group \( G_K \) acts on \( \pi_1(\mathbb{P}^1_K \setminus \{0, 1, \infty\}, \overrightarrow{01}) \) – the pro-\( p \) étale fundamental group. Let \( \mathbb{Q}_p\{\{X, Y\}\} \) be the \( \mathbb{Q}_p \)-algebra of non-commutative formal power series in two non-commuting variables \( X \) and \( Y \). Let \( E : \pi_1(\mathbb{P}^1_K \setminus \{0, 1, \infty\}, \overrightarrow{01}) \rightarrow \mathbb{Q}_p\{\{X, Y\}\} \) be the continuous multiplicative embedding given by \( E(x) = \exp X \) and \( E(y) = \exp Y \), where \( x \) and \( y \) are standard generators of \( \pi_1(\mathbb{P}^1_K \setminus \{0, 1, \infty\}, \overrightarrow{01}) \). For any \( \sigma \in G_K \) we define

\[
\tilde{f}_\gamma(\sigma) := \gamma^{-1} \cdot \sigma(\gamma) \in \pi_1(\mathbb{P}^1_K \setminus \{0, 1, \infty\}, \overrightarrow{01})
\]

and

\[
\Lambda_\gamma(\sigma) := E(\tilde{f}_\gamma(\sigma)) \in \mathbb{Q}_p\{\{X, Y\}\}.
\]

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In the special case of the path $\pi$ from $01$ to $10$, the element $f_\pi(\sigma)$ was studied by Ihara and his students (see [4] and more other papers), Deligne (see [1]), Grothendieck. The coefficients of the power series $\Lambda\pi(\sigma)$ are analogues of the multizeta numbers studied already by Euler. For an arbitrary path $\gamma$ the coefficients of the power series $\Lambda\gamma(\sigma)$ are analogues of values of iterated integrals evaluated at $z$.

Observe that
\[
\Lambda\gamma(\sigma) \equiv 1 + l_\gamma(z)(\sigma)X \mod I^2 + (Y)
\]
for a certain $l_\gamma(z)(\sigma) \in \mathbb{Z}_p$, where $I$ is the augmentation ideal of $\mathbb{Q}_p\{\{X,Y\}\}$ and $(Y)$ is the principal ideal generated by $Y$. Let us set
\[
\Delta\gamma(\sigma) := \exp(-l_\gamma(z)(\sigma)X) \cdot \Lambda\gamma(\sigma).
\]

One possible way to calculate (some) coefficients of the power series $\Lambda\pi(\sigma)$ and some other power series $\Lambda\gamma(\sigma)$ is to use symmetries of $\mathbb{P}_Q^1 \setminus \{0,1,\infty\}$, i.e. the so-called Drinfeld-Ihara relations (see [3] and [5]). For example in [11], we have calculated even polylogarithmic coefficients of the power series $\Lambda\pi(\sigma)$ using the symmetries of $\mathbb{P}_Q^1 \setminus \{0,1,\infty\}$.

In [8] the authors have constructed a measure on $\mathbb{Z}_p$ for any path $\gamma$ and expressed the $k$-th polylogarithmic coefficient of the power series $\log\Delta\gamma(\sigma)$ as integrals of the polynomial $x^{k-1}$ against this measure recovering the old result of O. Gabber (see [2]). Let us denote this measure by $K(z)_\gamma$.

Now we shall describe the main result of this note. Let $m$ be a positive integer not divisible by $p$. Let us set
\[
\xi_m = \exp\left(\frac{2\pi \sqrt{-1}}{m}\right).
\]

Let $0 < i < m$. Further we chose paths $\beta_i$ (resp. $\beta_{m-i}$) on $\mathbb{P}_Q^1 \setminus \{0,1,\infty\}$ from $01$ to $\xi_m$ (resp. $\xi_m^{-i}$) such that $l_{\beta_i}(\xi_m^i) = 0$ and $l_{\beta_{m-i}}(\xi_m^{m-i}) = 0$.

In [12] using the symmetry $\bar{z} \mapsto 1/\bar{z}$ of $\mathbb{P}_Q^1 \setminus \{0,1,\infty\}$ we have shown that the polylogarithmic coefficient in degree $k$ of the formal power series
\[
\log\Lambda_{\beta_{m-i}}(\sigma) + (-1)^i \log\Lambda_{\beta_i}(\sigma)
\]
is equal $\frac{B_k(\chi^i)(1 - \chi^k(\sigma))}{k!}$, where $B_k(X)$ is the $k$-th Bernoulli polynomial and $\chi : G_{\mathbb{Q}(\mu_m)} \to \mathbb{Z}_p^\times$ is the cyclotomic character (see [12, Theorem 10.2]). In this paper we shall calculate the same polylogarithmic coefficients using the measure
\[
K(\xi_m^{m-i})_{\beta_{m-i}} + i(K(\xi_m^i)_{\beta_i}),
\]
where $i$ is the complex conjugation acting on measures. To calculate these measures we use the symmetry $\bar{z} \mapsto 1/\bar{z}$ of the tower of coverings
\[
\mathbb{P}_Q^1 \setminus \{0,\infty\} \cup \mu_p \cdot \mathbb{P}_Q^1 \setminus \{0,1,\infty\}, \ \bar{z} \mapsto \bar{z}^p,
\]
of $\mathbb{P}_Q^1 \setminus \{0,1,\infty\}$. However in contrast with the calculations in [12] we need to work only with terms in degree 1. We show that the measure $K(\xi_m^{m-i})_{\beta_{m-i}} + i(K(\xi_m^i)_{\beta_i})$ is the sum of the Bernoulli measure $E_{1,\chi}$ (see [6, the formula E.1 on page 38]) and the measure we denote by $\mu_{\chi}(\xi_m^i/m)$. The definition of the measure $\mu_{\chi}(\xi_m^i/m)$ is very elementary and perhaps it is well known. From this it follows immediately the formula for the $k$-th polylogarithmic coefficient of the power series (1). The measure we got, allows to get the $p$-adic Hurwitz zeta functions as Mellin transform in the
same way as the $p$-adic $L$-functions are the Mellin transforms of the measure $\psi E_{1,c}$, where $\psi$ is a character on $\mathbb{Z}_p^\times$ (see [6, Chapter 4]).

1. An example of a measure on $\mathbb{Z}_p$

This section can be seen as an attempt to construct a measure on $\mathbb{Z}_p$ which to a subset $a + p^n\mathbb{Z}_p$ associates $1/p^n$. We found the measure in question studying Galois actions on torsors of paths (see section 3). The measure is elementary and we think that it should be known.

If $a \in \mathbb{Z}_p$ and $a = \sum_{i=0}^{\infty} \alpha_i p^i$ with $0 \leq \alpha_i \leq p - 1$ then we set

$$v_n(a) := \sum_{i=0}^{n} \alpha_i p^i \quad \text{and} \quad t_{n+1}(a) := \frac{a - v_n(a)}{p^{n+1}}.$$ 

Let us fix a positive integer $m > 1$. For $k \in \mathbb{Q}^\times$, $k = \frac{a}{b}$ with $a, b \in \mathbb{Z}$ and $(b, m) = 1$ we define

$$[k]_m \in \mathbb{N}$$

by the following two conditions

$$0 \leq [k]_m < m \quad \text{and} \quad b[k]_m \equiv a \mod m.$$ 

Let us assume that $p$ does not divide $m$. Let $i$ be such that $0 < i < m$. Observe that

$$[p^{-r}[ip^{-n}]]_m = [ip^{-(n+r)}]_m.$$ 

We define a sequence of integers

$$(k_r(i))_{r \in \mathbb{N}}$$

by the equalities

$$p[ip^{-r}]_m = [ip^{-(r-1)}]_m + k_{r-1}(i)m.$$ 

Observe that

$$0 < \frac{[ip^{-(r-1)}]_m}{m} < 1 \quad \text{and} \quad 0 < \frac{p[ip^{-r}]}{m} < p.$$ 

Hence it follows that

$$0 \leq k_r(i) \leq p - 1$$

for all $r \geq 0$. Applying successively the formula (3) we get

$$p^n[ip^{-n}]_m = i + \left(\sum_{\alpha=0}^{n-1} k_{\alpha}(i)p^\alpha\right)m.$$ 

It follows from (4) that

$$\frac{-i}{m} = \sum_{\alpha=0}^{\infty} k_{\alpha}(i)p^\alpha$$

and

$$\frac{i}{m} = 1 + \sum_{\alpha=0}^{\infty} (p - 1 - k_{\alpha}(i))p^\alpha.$$ 

Another consequence of (4) is the equality

$$t_{\alpha}\left(\frac{-i}{m}\right) = \frac{-[ip^{-n}]_m}{m}.$$
For any positive integer \(a\) such that \(0 \leq a < p^n\) we set
\[
\delta_n(a) := \begin{cases} 
-1 & \text{if } a \geq 1 + \sum_{\alpha=0}^{n-1}(p-1-k_\alpha(i))p^\alpha, \\
0 & \text{if } a < 1 + \sum_{\alpha=0}^{n-1}(p-1-k_\alpha(i))p^\alpha.
\end{cases}
\]

**Definition-Proposition 1.1.** The function from the open-closed subsets of \(\mathbb{Z}_p\) to \(\mathbb{Z}_p\) defined by the formula
\[
\mu\left(\frac{i}{m}\right)(a + p^n\mathbb{Z}_p) := \left[\frac{ip-n}{m}\right] + \delta_n(a)
\]
for \(0 \leq a < p^n\) is a measure.

**Proof.** Let \(0 \leq a < p^n\). We have
\[
\sum_{b=0}^{p-1} \mu\left(\frac{i}{m}\right)(a + bp^n + p^n+1\mathbb{Z}_p) = \sum_{b=0}^{p-1} \left(\frac{[ip-(n+1)]}{m} + \delta_{n+1}(a + bp^n)\right) = 
\]
\[
\left[\frac{ip-n}{m}\right] + \sum_{b=0}^{p-1} \delta_{n+1}(a + bp^n) = \left[\frac{ip-n}{m}\right] + \sum_{b=0}^{p-1} \delta_{n+1}(a + bp^n)
\]
by the equality(3). Observe that
\[
\sum_{b=0}^{p-1} \delta_{n+1}(a + bp^n) := \begin{cases} 
-k_n(i) - 1 & \text{if } a \geq 1 + \sum_{\alpha=0}^{n-1}(p-1-k_\alpha(i))p^\alpha, \\
-k_n(i) & \text{if } a < 1 + \sum_{\alpha=0}^{n-1}(p-1-k_\alpha(i))p^\alpha.
\end{cases}
\]
Hence finally we get \(\sum_{b=0}^{p-1} \mu\left(\frac{i}{m}\right)(a + bp + p^n+1\mathbb{Z}_p) = \left[\frac{ip-n}{m}\right] + \delta_n(a) = \mu\left(\frac{i}{m}\right)(a + p^n\mathbb{Z}_p).\)

**Proposition 1.2.** For \(k \geq 1\) we have
i)
\[
\int_{\mathbb{Z}_p} x^{k-1} d\mu\left(\frac{i}{m}\right)(x) = \frac{1}{k} \left( B_k\left(\frac{i}{m}\right) - B_k \right),
\]

ii)
\[
\int_{\mathbb{Z}_p} x^{k-1} d\mu\left(\frac{i}{m}\right)(x) = \frac{1}{k} \left( B_k\left(\frac{i}{m}\right) - B_k \right) - \frac{p^{k-1}}{k} \left( B_k\left(\left[\frac{ip-n}{m}\right]\right) - B_k \right).
\]

**Proof.** First we shall prove the formula i). Let us calculate the Riemann sum
\[
\sum_{\alpha=0}^{p^n-1} \alpha^{k-1} \mu\left(\frac{i}{m}\right)(\alpha + p^n\mathbb{Z}_p) = \sum_{\alpha=0}^{p^n-1} \alpha^{k-1} \left(\frac{[ip-n]}{m}\right) + \delta_n(\alpha) = 
\]
\[
\left[\frac{ip-n}{m}\right] \sum_{\alpha=0}^{p^n-1} \alpha^{k-1} - \sum_{\alpha=0}^{p^n-1} \alpha^{k-1} + \sum_{\alpha=0}^{v_{n-1}(\frac{i}{m})-1} \alpha^{k-1}.
\]
Observe that
\[
\sum_{\alpha=0}^{v_{n-1}(\frac{i}{m})-1} \alpha^{k-1} = \frac{1}{k} \left( B_k(v_{n-1}(\frac{i}{m})) - B_k \right)
\]
and it tends to \(\frac{1}{k} \left( B_k\left(\frac{i}{m}\right) - B_k \right)\) if \(n\) tends to \(\infty\). Hence the formula i) of the proposition follows because \(\sum_{\alpha=0}^{p^n-1} \alpha^{k-1}\) tends to \(0\) if \(n\) tends to \(\infty\).
Observe that
\[
\int_{\mathbb{Z}_p} x^{k-1} d\mu \left( \frac{i}{m} \right)(x) = \int_{\mathbb{Z}_p} x^{k-1} d\mu \left( \frac{i}{m} \right)(x) - \int_{\mathbb{Z}_p} x^{k-1} d\mu \left( \frac{i}{m} \right)(x).
\]
We shall calculate Riemann sums for the integral \( \int_{\mathbb{Z}_p} x^{k-1} d\mu \left( \frac{i}{m} \right)(x) \). We have
\[
\sum_{\alpha=0}^{p^\alpha - 1} \frac{p^\alpha - 1}{\mu} \mu (p\alpha + p^n \mathbb{Z}_p) = \sum_{\alpha=0}^{p^\alpha - 1} p^k \alpha^{k-1} \frac{[ip^{n-1}]m}{m} + \sum_{\alpha=0}^{p^\alpha - 1} p^k \alpha^{k-1} \delta_{n+1}(p\alpha).
\]
The first sum tend to 0 if \( n \) tends to \( \infty \). Observe that
\[
\sum_{\alpha=0}^{p^\alpha - 1} p^k \alpha^{k-1} \delta_{n+1}(p\alpha) = \sum_{0<\alpha<p\alpha, \alpha \geq v_n \left( \frac{1}{m} \right)} p^k \alpha^{k-1} (-1) = \sum_{\alpha=0}^{p^\alpha - 1} p^k \alpha^{k-1} + \sum_{0<\alpha<p\alpha, \alpha < v_n \left( \frac{1}{m} \right)} p^k \alpha^{k-1}.
\]
Let \( 0 \leq \beta_0 < p \) be such that \( v_n \left( \frac{1}{m} \right) \equiv \beta_0 \) modulo \( p \). Then
\[
v_n - 1 \left( \frac{[ip^{n-1}]m}{m} \right) = \begin{cases} 1 + \frac{1}{p} (v_n \left( \frac{1}{m} \right) - \beta_0) & \text{if } \beta_0 \neq 0, \\ \frac{1}{p} v_n \left( \frac{1}{m} \right) & \text{if } \beta_0 = 0. \end{cases}
\]
Hence it follows that
\[
\sum_{0<\alpha<p\alpha, \alpha \geq v_n \left( \frac{1}{m} \right)} p^k \alpha^{k-1} = p^k - \sum_{\alpha=0}^{p^\alpha - 1} v_n - 1 \left( \frac{[ip^{n-1}]m}{m} \right).
\]
If \( n \) tends to \( \infty \) the last sum tends to \( p^k \frac{1}{k} (B_k \left( \frac{[ip^{n-1}]m}{m} \right) - B_k) \). Hence the proof of the formula ii) is finished.

If \( c \in \mathbb{Z}_p \setminus \mu_{p-1} \) we define

(6) \[
\mu_c \left( \frac{i}{m} \right) := \mu \left( \frac{i}{m} \right) - c \mu \left( \frac{i}{m} \right) \circ c^{-1}.
\]
Then we have

(7) \[
\frac{1}{1 - c^k} \int_{\mathbb{Z}_p} x^{k-1} d\mu_c \left( \frac{i}{m} \right)(x) = \frac{1}{k} (B_k \left( \frac{i}{m} \right) - B_k).
\]

**Corollary 1.3.** Let \( P : \mathbb{Z}_p[[\mathbb{Z}_p]] \to \mathbb{Z}_p[[T]] \) be the Iwasawa isomorphism given by \( P(1) = 1 + T \). Then

\[
P(\mu \left( \frac{i}{m} \right))(T) = \frac{(1 + T)^{\frac{1}{m}} - 1}{T}
\]
and
\[
P(\mu_c \left( \frac{i}{m} \right)) = \frac{(1 + T)^{\frac{1}{m}} - 1}{T} - c \left( (1 + T)^{\frac{1}{m}} - 1 \right) \frac{1}{(1 + T)^c - 1}.
\]

**Proof.** The power series \( P(\mu \left( \frac{i}{m} \right))(\exp X - 1) \) is equal \( \sum_{k=0}^{\infty} \left( \int_{\mathbb{Z}_p} x^{k} d\mu \left( \frac{i}{m} \right)(x) \right) X^k \).
Hence by the point i) of Proposition 1.2 it is equal

\[
\sum_{k=0}^{\infty} \frac{1}{(k+1)!} (B_{k+1} \left( \frac{i}{m} \right) - B_{k+1}) X^k.
\]
It follows from the definition of the Bernoulli numbers and the Bernoulli polynomials that this power series is equal \( \exp \frac{X}{1 - X} \). Replacing \( X \) by \( 1 + T \) we get the power series \( P(\mu(\frac{1}{m}))(T) \).

We denote by
\[ \omega : \mathbb{Z}_p^\times \to \mu_{p-1} \subset \mathbb{Z}_p^\times \]
the Teichmüller character. For \( x \in \mathbb{Z}_p^\times \) we set
\[ [x] := x\omega(x)^{-1}. \]

Let us define
\[ \hat{H}_p(1 - s, \omega^b, \frac{i}{m}) := \int_{\mathbb{Z}_p^\times} [x]^s x^{-1} \omega(x)^b d\mu(\frac{i}{m})(x). \]

**Proposition 1.4.** Let \( k \equiv b \mod p - 1 \). Then
\[ \hat{H}_p(1 - k, \omega^b, \frac{i}{m}) = \frac{1}{k} (B_k(\frac{i}{m}) - B_k) - \frac{p^{k-1}}{k} (B_k(\frac{[ip]^{-1}}{m}) - B_k). \]

**Proof.** We have
\[ \hat{H}_p(1 - k, \omega^b, \frac{i}{m}) = \int_{\mathbb{Z}_p^\times} [x]^k x^{-1} \omega(x)^b d\mu(\frac{i}{m})(x) = \int_{\mathbb{Z}_p^\times} x^{k-1} d\mu(\frac{i}{m})(x). \]

Hence the proposition follows from the formula ii) of Proposition 1.2. \( \square \)

**Remark 1.5.** A function closely related to our function \( \hat{H}_p(1 - s, \omega^b, \frac{i}{m}) \) appears in a paper of Shiratani (see [10, Theorem 1, case \( p \nmid f \)]).

2. Action of the complex conjugation on measures

We define an action of \( \mathbb{Z}_p^\times \) on the group ring \( \mathbb{Z}_p[\mathbb{Z}_p] \) by the formula
\[ \alpha(\sum_{i=1}^n a_i(x_i)) = \alpha \sum_{i=1}^n a_i(\alpha^{-1} x_i) \]
and we extend by continuity to the action of \( \mathbb{Z}_p^\times \) on \( \mathbb{Z}_p[[\mathbb{Z}_p]] \). The action of \(-1 \in \mathbb{Z}_p^\times \) we denote by \( \iota \). Then
\[ \mathbb{Z}_p[[\mathbb{Z}_p]] = \mathbb{Z}_p[[\mathbb{Z}_p]]^+ \oplus \mathbb{Z}_p[[\mathbb{Z}_p]]^- , \]
where \( \iota \) acts on \( \mathbb{Z}_p[[\mathbb{Z}_p]]^+ \) (resp. on \( \mathbb{Z}_p[[\mathbb{Z}_p]]^- \)) as the identity (resp. as the multiplication by \(-1 \)). For any \( \mu \in \mathbb{Z}_p[[\mathbb{Z}_p]] \) we have the decomposition
\[ \mu = \mu^+ + \mu^- , \]
where \( \mu^+ = \frac{1}{2}(\mu + \iota(\mu)) \in \mathbb{Z}_p[[\mathbb{Z}_p]]^+ \) and \( \mu^- = \frac{1}{2}(\mu - \iota(\mu)) \in \mathbb{Z}_p[[\mathbb{Z}_p]]^- \). Observe that
\[ \int_{\mathbb{Z}_p} x^{k-1} d\mu = (-1)^k \int_{\mathbb{Z}_p} x^{k-1} d\mu . \]

Hence it follows
\[ \int_{\mathbb{Z}_p} x^{k-1} d\mu^+ := \begin{cases} 0 & \text{for } k \text{ odd}, \\ \int_{\mathbb{Z}_p} x^{k-1} d\mu & \text{for } k \text{ even} \end{cases} \]
and

\[(10) \quad \int_{\mathbb{Z}_p} x^{k-1} d\mu^- := \begin{cases} \int_{\mathbb{Z}_p} x^{k-1} d\mu & \text{for } k \text{ odd}, \\ 0 & \text{for } k \text{ even}. \end{cases} \]

In [12, Proposition 10.5] we have shown that

\[(11) \quad \int_{\mathbb{Z}_p} x^{k-1} d(K(\xi_m^{-i}) + K(\xi_m^i)) = \frac{1}{k} B_k(\frac{i}{m})(1 - \chi^k) \text{ for } k \text{ even} \]

and

\[(12) \quad \int_{\mathbb{Z}_p} x^{k-1} d(K(\xi_m^{-i}) - K(\xi_m^i)) = \frac{1}{k} B_k(\frac{i}{m})(1 - \chi^k) \text{ for } k \text{ odd}. \]

Hence it follows from (9) and (10) that

\[(13) \quad \int_{\mathbb{Z}_p} x^{k-1} d((K(\xi_m^{-i}) + K(\xi_m^i))^+ + (K(\xi_m^{-i}) - K(\xi_m^i))^-) = \frac{1}{k} B_k(\frac{i}{m})(1 - \chi^k) \text{ for } k \geq 1. \]

Observe that

\[(K(\xi_m^{-i}) + K(\xi_m^i))^+ + (K(\xi_m^{-i}) - K(\xi_m^i))^-) = K(\xi_m^{-i}) + \iota(K(\xi_m^i)). \]

Hence we get

\[(14) \quad \int_{\mathbb{Z}_p} x^{k-1} d(K(\xi_m^{-i}) + \iota(K(\xi_m^i))) = \frac{1}{k} B_k(\frac{i}{m})(1 - \chi^k) \text{ for } k \geq 1. \]

The proof of the formulas (11) and (12) given in [12] is based on the symmetry $z \mapsto 1/z$ of $\mathbb{P}_1^1 \setminus \{0,1,\infty\}$ and the study of the polylogarithmic coefficients (at $Y^{X^{k-1}}$) of the power series $\Lambda_{\beta_i}(\sigma)$ and $\Lambda_{\beta_m-i}(\sigma)$. Recently, H. Nakamura (see [7]) got these formulas using directly the inversion formula from [9, section 6.3].

In this paper we calculate explicitly the measure $K(\xi_m^{-i}) + \iota(K(\xi_m^i))$. We use also the symmetry $z \mapsto 1/z$ of the tower of coverings

\[\mathbb{P}_1^1 \setminus \{0,1,\infty\} \cup \mu_{p^n} \rightarrow \mathbb{P}_1^1 \setminus \{0,1,\infty\}, \quad z \mapsto z^{p^n}\]

but only in degree 1.

The third possible method to calculate the measure $K(\xi_m^{-i}) + \iota(K(\xi_m^i))$ is to use the explicit formula for measures $K(z)$ (see [8, Proposition 3]). Compare the three different proofs of Proposition 5.13 in [9]. Two proofs are given in [9] and the third one in [12] (the second proof of Lemma 4.1.)

3. Measures associated with roots of unity

We set

\[\xi_r := \exp(\frac{2\pi \sqrt{-1}}{r})\]

for a natural number $r$. Let us set

\[V_n := \mathbb{P}_1^1 \setminus \{0,\infty\} \cup \mu_{p^n}\].

We recall that $\pi_1(V_n,0)$ - pro-$p$ étale fundamental group - is free on generators $x_n$ - loop around 0 - and $y_{n,i}$ - loops around $\xi_{p^n}^i$ for $0 \leq i < p^n$. 
For each $0 < i < m$, let $\alpha_i$ be a path on $V_0 = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$ from $0$ to $\xi^i_m$ which is the composition of an arc from $0$ to $0\xi^i_m$ in an infinitesimal neighbourhood of $0$ followed by the canonical path (straight line) from $0\xi^i_m$ to $\xi^i_m$.

Let us set

$$\beta_i := \alpha_i \cdot x^{-\frac{i}{m}}.$$  

Observe that $l(\xi^i_{m})\beta_i = 0$. If we regard the path $\alpha_i$ as the path on $V$ then we denote it by

$$n\alpha_i.$$  

Then

$$\beta_i := n\alpha_i \cdot x^{-\frac{i}{m}}$$  

is also a path on $V_n$. Let

$$\tilde{\beta}_i^n$$  

(resp. $\tilde{\alpha}_i^n$) be the lifting of $\beta_i$ (resp. $\alpha_i$) to $V_n$ starting from $0\tilde{\beta}_i^n$. Let $0 \leq j < p^n$. We denote by $s^n_i$ a lifting of $x^n_j$ to $V_n$ starting from $0\tilde{\beta}_i^n$. Observe that $s^n_i$ is a path on $V_n$ from $0\tilde{\beta}_i^n$ to $0\tilde{\beta}_i^n P^n$. Let $0 < \xi$.

**Lemma 3.1.** We have

$$\tilde{\beta}_i^n = n\tilde{\beta}_i^{[ip^n]_m} = n\alpha_i^{[ip^n]_m} \cdot x_n^{-\frac{[ip^n]_m}{m}}.$$  

**Proof.** Observe that the lifting of $x^{-\frac{i}{m}}$ to $V_n$ is equal $s^n_{v^n_{n-1}(-\frac{i}{m})_m} x^n_{v^n_{n-1}(-\frac{i}{m})}$.

The lifting of $\alpha_i$ to $V_n$ is a path (an arc) from $0$ to $\xi := 0\xi^i_{m} P^n$ in the positive sense composed with the canonical path from $\xi$ to $\xi^i_{m} P^n$. Hence the lifting of $\beta_i$ is the composition of $s^n_{v^n_{n-1}(-\frac{i}{m})_m} x^n_{v^n_{n-1}(-\frac{i}{m})}$ with the lifting of $\alpha_i$ multiplied by $\xi^{v^n_{n-1}(-\frac{i}{m})}$. We have

$$\xi^{v^n_{n-1}(-\frac{i}{m})_m} \xi^{v^n_{n-1}(-\frac{i}{m}) + i} = \xi^{v^n_{n-1}(-\frac{i}{m}) + i}.$$  

Observe that $0 \leq v^n_{n-1}(-\frac{i}{m})_m + i < p^n m$ and that $p^n$ divides $v^n_{n-1}(-\frac{i}{m})_m + i$. Moreover we have $\frac{v^n_{n-1}(-\frac{i}{m})_m + i}{p^n} \cdot p^n \equiv i \pmod{m}$. Hence it follows that

$$\frac{v^n_{n-1}(-\frac{i}{m})_m + i}{p^n} = [ip^n]_m.$$  

Therefore we get

$$-\frac{[ip^n]_m}{m} = -\frac{i}{m} + \frac{m}{m} = t_n(-\frac{i}{m}).$$  

Hence it follows that the lifting of $\beta_i$ is $n\alpha_i^{[ip^n]_m} \cdot x_n^{-\frac{[ip^n]_m}{m}}$. \hfill $\square$

To simplify the notation we set

$$r_n = [ip^n]_m \quad \text{and} \quad v^n_{n-1} = v^n_{n-1}(-\frac{i}{m}).$$  

Then we have

$$\tilde{\beta}_i^n = n\alpha_r \cdot x_n^\frac{i}{m} \quad \text{and} \quad \tilde{\beta}_m = n\alpha_m - r_n \cdot x_n^\frac{i}{m} - 1.$$
Let $h : V_n \to V_n$ be given $1 \to 1/3$. Let $p_n$ be the canonical path from $1 \to V_n$, $t_n$ a path from $10$ to $1 \times (\text{half circle in the positive sense in an infinitesimal neighbourhood of } 1)$ and $q_n = h(p_n)$. We set

$$\Gamma_n := q_n \cdot t_n \cdot p_n.$$ 

**Lemma 3.2.** We have

$$\rho_{m-i} = h(\rho_{m}) \cdot \Gamma_n \cdot z_{m} \cdot x_{n} \cdot y_{n-1} \cdot \ldots \cdot y_{n-v_{n-1}} \cdot x_{n}^{\kappa} - 1$$

in $\pi_1(V_n, 01)$.

**Proof.** One checks that $n \alpha_{m - r_n} = h(n \alpha_{r_n}) \cdot \Gamma_n \cdot x_{n} \cdot y_{n-1} \cdot \ldots \cdot y_{n-v_{n-1}}$. The formula of the lemma follows from Lemma 3.1. \hfill $\square$

**Lemma 3.3.** Let $\sigma \in G_{Q(\mu_n)}$. Then writing additively we have

$$\Gamma_n(\sigma) = \sum_{k=0}^{p^n-1} E_{0, \chi(\sigma)}^{(n)}(k)y_{n,k} \bmod (\pi_1(V_n, 01), \pi_1(V_n, 01)).$$

**Proof.** See the proof of Lemma 4.1 in [12] or the second proof of Proposition 5.13 in [9]. \hfill $\square$

It follows from Lemma 3.2 that

$$\Gamma_{m-i}^{(n)}(\sigma) = \Gamma_n \cdot h(\Gamma_{m-i}^{(n)}(\sigma)) \cdot \Gamma_n \cdot \Gamma_{m-i}^{(n)}(\sigma),$$

modulo $\pi_1(V_n, 01), \pi_1(V_n, 01))$. Hence writing the result additively we get

$$\sum_{k=0}^{p^n-1} K^{(n)}(\xi^{m-i}(\sigma)) (k) y_{n,k} = \sum_{k=0}^{p^n-1} K^{(n)}(\xi^{m-i}(\sigma)) (k) y_{n,k} + \sum_{k=0}^{p^n-1} E_{1, \chi(\sigma)}^{(n)}(k) y_{n,k} +$$

$$\sum_{k=0}^{p^n-1} (1 - \chi(\sigma)) \frac{[i p^n - n]}{m} y_{n,k} - \sum_{j=1}^{v_{n-1}(-\frac{i}{m})} y_{n,-j} + \chi(\sigma) \sum_{j=1}^{v_{n-1}(-\frac{i}{m})} y_{n,-[j \chi(\sigma)]p^n}$$

modulo $\pi_1(V_n, 01), \pi_1(V_n, 01))$. Observe that $v_{n-1}(\frac{i}{m}) = p^n - v_{n-1}(-\frac{i}{m})$. Hence the last two sums we can rewrite in the form

$$\sum_{j=v_{n-1}(-\frac{i}{m})}^{p^n-1} y_{n,j} + \chi(\sigma) \sum_{j=v_{n-1}(\frac{i}{m})}^{p^n-1} y_{n,[j \chi(\sigma)]p^n}.$$

Comparing coefficients at $y_{n,k}$ we get for $0 \leq k < p^n$

$$K^{(n)}(\xi^{m-i}(\sigma)) (k) - K^{(n)}(\xi^{m-i}(\sigma)) (-k) =$$

$$E_{0, \chi(\sigma)}^{(n)}(k) + \frac{[i p^n - n]}{m} \delta_n(k) - \chi(\sigma) \frac{[i p^n - n]}{m} + \chi(\sigma) \delta_n([\chi(\sigma) - 1]k_{p^n}) =$$

$$E_{1, \chi(\sigma)}^{(n)}(k) + \mu_{\chi(\sigma)}(\frac{i}{m})(k)$$

by the definition of the measure $\mu_{\chi(\sigma)}(\frac{i}{m})$. 

\hfill $\square$
Theorem 3.5. Let \( m \) be a positive integer not divisible by \( p \) and let \( 0 < i < m \). Then we have
\[
K(\xi_m^{-i})(\sigma) + \iota(K(\xi_m^i)(\sigma)) = E_{1,\chi(\sigma)} + \mu_{\chi(\sigma)}(\frac{i}{m}).
\]

Proof. The theorem follows from the formula (15).
\( \square \)

Corollary 3.6. Let \( \sigma \in G_{Q(\mu_m)} \) be such that \( \chi(\sigma)^{p-1} \neq 1 \). Then we have

i) \[
\frac{1}{1 - \chi(\sigma)^k} \int_{\mathbb{Z}_p} x^{k-1} d(K(\xi_m^{-i})(\sigma) + \iota(K(\xi_m^i)(\sigma))) = \frac{B_k(\frac{i}{m})}{k},
\]

ii) \[
P(K(\xi_m^{-i})(\sigma) + \iota(K(\xi_m^i)(\sigma)))(T) = \frac{(1 + T)^{\frac{1}{m}}}{(1 + T)^{\chi(\sigma)^{\frac{1}{m}}}} - \frac{\chi(\sigma)(1 + T)^{\chi(\sigma)^{\frac{1}{m}}}}{(1 + T)^{\chi(\sigma) - 1}}.
\]

Proof. The point i) of the corollary follows from Theorem 3.5 and the formula (7). The point ii) follows immediately from Corollary 1.8 and the equality \( P(E_{1,\chi(\sigma)})(T) = \frac{1}{T} - \frac{\chi(\sigma)^i}{(1 + T)^{\chi(\sigma) - 1}} \).
\( \square \)

Now we define
\[
L^B(1 - s, (\xi_m^{-i}) + \iota(\xi_m^i); \sigma) := \frac{1}{1 - \omega(\chi(\sigma))^2[\chi(\sigma)]^k} \int_{\mathbb{Z}_p^*} [x]^y x^{-1} \omega(x)^\beta d((K(\xi_m^{-i}) + \iota(K(\xi_m^i)))(\sigma)).
\]

Theorem 3.7. Let \( \sigma \in G_{Q(\mu_m)} \) be such that \( \chi(\sigma)^{p-1} \neq 1 \).

i) Let \( k \equiv \beta \) modulo \( p - 1 \). Then
\[
L^B(1 - k, (\xi_m^{-i}) + \iota(\xi_m^i); \sigma) = \frac{1}{k} B_k(\frac{i}{m}) - p^{-1} \frac{1}{k} B_k([ip^{-1}]m).
\]

ii) Let \( \sigma, \sigma_1 \in G_{Q(\mu_m)} \) be such that \( \chi(\sigma)^{p-1} \neq 1 \) and \( \chi(\sigma_1)^{p-1} \neq 1 \). Then
\[
L^B(1 - s, (\xi_m^{-i}) + \iota(\xi_m^i); \sigma) = L^B(1 - s, (\xi_m^{-i}) + \iota(\xi_m^i); \sigma_1),
\]
i.e. the function \( L^B(1 - s, (\xi_m^{-i}) + \iota(\xi_m^i); \sigma) \) does not depend on \( \sigma \).

Proof. For \( k \equiv \beta \) modulo \( p - 1 \) we have
\[
L^B(1 - k, (\xi_m^{-i}) + \iota(\xi_m^i); \sigma) = \frac{1}{1 - \chi(\sigma)^k} \int_{\mathbb{Z}_p^*} x^{k-1} d(\mu_{\chi(\sigma)}(\frac{i}{m}) + E_{1,\chi(\sigma)})
\]
by Theorem 3.5. It follows from [6, Theorem 2.3] that \( \frac{1}{\chi(\sigma)^{k-1}} \int_{\mathbb{Z}_p^*} x^{k-1} dE_{1,\chi(\sigma)} = -\frac{1}{k} B_k \). The “periodicity” property \( E_{1,\chi(\sigma)}^{(n)}(i) = E_{1,\chi(\sigma)}^{(n+1)}(pi) \) of the measure \( E_{1,\chi(\sigma)} \) implies that
\[
\frac{1}{1 - \chi(\sigma)^k} \int_{\mathbb{Z}_p^*} x^{k-1} dE_{1,\chi(\sigma)} = (1 - p^{-1}) \frac{1}{k} B_k.
\]

Integrating the function \( x^{k-1} \) against the measure \( \mu_{\chi(\sigma)}(\frac{i}{m}) \) we get
\[
\frac{1}{\chi(\sigma)^k - 1} \int_{\mathbb{Z}_p^*} x^{k-1} d\mu_{\chi(\sigma)}(\frac{i}{m})(x) =
\]
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\[
\frac{1}{\chi(\sigma)^k - 1} \left( \int_{\mathbb{Z}_p^+} x^{k-1} d\mu(\frac{i}{m})(x) - \int_{\mathbb{Z}_p^+} x^{k-1} d(\chi(\sigma)\mu(\frac{i}{m}) \circ \chi(\sigma)^{-1})(x) \right).
\]

Observe that \(\int_{\mathbb{Z}_p^+} x^{k-1} d(\chi(\sigma)\mu(\frac{i}{m}) \circ \chi(\sigma)^{-1})(x) = \chi(\sigma)^k \int_{\mathbb{Z}_p^+} y^{k-1} d\mu(\frac{y}{m})(y)\) if we set \(\chi(\sigma)y = x\). It follows from Proposition 1.9 that

\[
\frac{1}{\chi(\sigma)^k - 1} \int_{\mathbb{Z}_p^+} x^{k-1} d\mu(\frac{i}{m})(x) = \frac{1}{k} \left( B_k(\frac{i}{m}) - B_k \right) - p^{k-1} \frac{1}{k} \left( B_k(\frac{[ip-1]m}{m}) - B_k \right).
\]

After the addition of (16) and (17) we get the point i) of the theorem.

Concerning the point ii) observe that the functions \(L^\beta (1-s, (\xi_m^{-i}) + i(\xi_m^i); \sigma)\) and \(L^\beta (1-s, (\xi_m^{-i}) + i(\xi_m^i); \sigma_1)\) coincide for \(k \equiv \beta\) modulo \((p-1)\). Hence these functions are equal because they are equal on a dense subset of \(\mathbb{Z}_p^n\). \(\square\)

References


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