On adelic Hurwitz zeta measures

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Abstract. In this paper we construct a \( \hat{\mathbb{Z}} \)-valued measure on \( \hat{\mathbb{Z}} \) which interpolates \( p \)-adic Hurwitz zeta functions for all \( p \).

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1. Introduction

Let \( m \geq 1 \), \( 0 < a < m \) be integers such that \( a \) is prime to \( m \), and let \( p \) be a rational prime. Set \( q := 4 \), \( q := p \) according to whether \( p = 2 \) or \( p > 2 \) respectively, and \( e := |(\mathbb{Z}/q\mathbb{Z})^\times| \). When \( p \nmid m \), let \( \langle ap^{-1} \rangle \) denote the least positive integer such that \( \langle ap^{-1} \rangle \equiv a \mod m \). Define the Bernoulli polynomials \( B_k(T) \) \((k \in \mathbb{N})\) by

\[
\sum_{k=0}^{\infty} B_k(T) \frac{x^k}{k!} = \frac{e^{xT} - e^T}{e^T - 1}
\]

and set the Bernoulli numbers \( B_k := B_k(0) \).

In [Sh], Shiratani constructed \( p \)-adic Hurwitz zeta functions \( \zeta_{sh}^p(s; a, m) \) \((s \in \mathbb{Z}_p, s \neq 1)\) characterized by the interpolation property:

\[
\zeta_{sh}^p(1-k; a, m) = \begin{cases} 
-m_k^{-1}B_k(\frac{a}{m}), & (p \mid m); \\
-m_k^{-1}B_k(\frac{a}{m}) + p^{-1}m_k^{-1}B_k(\frac{(ap^{-1})^{-1}}{m}), & (p \nmid m)
\end{cases}
\]

for all integers \( k > 1 \) with \( k \equiv 0 \mod e \). In [W3], assuming \( p \nmid m \), the second author introduced a \( p \)-adic Hurwitz \( L \)-function \( L_p^\beta(s; a, m) \) for \( \beta \in (\mathbb{Z}/e\mathbb{Z}) \) which satisfies

\[
L_p^\beta(1-k; a, m) = \frac{1}{k}B_k(\frac{a}{m}) - \frac{p^{-1}m_k^{-1}B_k(\langle ap^{-1} \rangle^{-1})}{m}
\]

for all integers \( k > 1 \) with \( k \equiv \beta \mod e \) by using certain \( p \)-adic measures arising in the study of Galois actions on paths on \( \mathbb{P}^1 - \{0, 1, \infty\} \) (see also [W4]). The purpose of this paper is to complete the construction to include the case \( p \mid m \) and to lift it over \( \hat{\mathbb{Z}} = \varprojlim_N(\mathbb{Z}/N\mathbb{Z}) \).

Throughout this paper, we fix an embedding of \( \overline{\mathbb{Q}} \) into \( \mathbb{C} \). For any subfield \( F \subset \mathbb{C} \), denote by \( G_F \) the absolute Galois group \( \text{Gal}(\overline{F}/F) \).

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Theorem 1.1. Let $m$ and $a$ be mutually prime integers with $m > 1$, $0 < a < m$. Then, for every $\sigma \in G_{\mathbb{Q}(\mu_m)}$, there exists a certain measure $\hat{\zeta}_{a,m}(\sigma)$ in $\hat{\mathbb{Z}}[[\hat{\mathbb{Z}}]]$ such that for every prime $p$, its image $\zeta_{p,a,m}(\sigma)$ in $\mathbb{Z}_p[[\mathbb{Z}_p]]$ has the following integration properties over $\mathbb{Z}_p^*$:

$$\int_{\mathbb{Z}_p^*} b^{k-1} d\hat{\zeta}_{p,a,m}(\sigma)(b) = \begin{cases} (1 - \chi_p(\sigma)^k) \cdot m^{k-1} \cdot \frac{1}{k} B_k\left(\frac{a}{m}\right) & (p \mid m); \\
(1 - \chi_p(\sigma)^k) \cdot m^{k-1} \left(\frac{1}{k} B_k\left(\frac{a}{m}\right) - \frac{p^{k-1}}{k} B_k\left(\frac{(ap-1)}{m}\right)\right) & (p \nmid m)
\end{cases}$$

for all integers $k \geq 1$, where $\chi_p : G_{\mathbb{Q}} \to \mathbb{Z}_p^*$ denotes the $p$-adic cyclotomic character, and $\langle ap^{-1} \rangle$ represents the least positive integer such that $\langle ap^{-1} \rangle p \equiv a \mod m$.

Remark 1.2. Note that, in the above theorem, the case $m = 1$ is excluded. In fact, the case $m = a = 1$ corresponds to the $\hat{\mathbb{Z}}$-zeta function treated in [W2]. This separation of treatment is necessary for the appearance of tangential base point $\mathbb{Z}$ in the construction of measure, which causes replacements of both $B_k\left(\frac{a}{m}\right)$, $B_k\left(\frac{(ap-1)}{m}\right)$ of RHS by $B_k(1)$.

Remark 1.3. More generally, we construct the measure $\hat{\zeta}_{a,m}(\sigma) \in \hat{\mathbb{Z}}[[\hat{\mathbb{Z}}]]$ for $m > 1$ and $m \nmid a$ which satisfies the above integration property for all primes $p|m$ with $p \nmid a$. (Cf. Remark 5.6.)

Remark 1.4. Using any $\sigma \in G_{\mathbb{Q}(\mu_m)}$ with $\chi_p(\sigma)^e \neq 1$, we obtain from $\hat{\zeta}_{p,a,m}$ a set of $p$-adic Hurwitz functions $\{L_p^{[\beta]}(s,a,m)\}_{\beta \in (\mathbb{Z}/e\mathbb{Z})^*}$ by the standard integral

$$L_p^{[\beta]}(s,a,m) = \frac{1}{1 - \omega(\chi_p(\sigma))^{\beta}} \frac{[\chi_p(\sigma)]^{1-s}}{1-s} \int_{\mathbb{Z}_p^*} [b]^{1-s} b^{-1} \omega(b)^{\beta} d\hat{\zeta}_{p,a,m}(\sigma)(b)$$

where $\omega : \mathbb{Z}_p^* \to \mu_e$ is the Teichmüller character, and for every $b \in \mathbb{Z}_p^*$, $[b] \in 1 + q\mathbb{Z}_p$ is defined by $b = [b] \omega(b)$. Note that the above integral converges in $s \in \mathbb{Z}_p$ except when it has a pole at $s = 1$ in the case $\beta \equiv 0 \mod e)$. It follows from Theorem 1.1 that, for each $\beta \in \mathbb{Z}/e\mathbb{Z}$, the $L$-function $L_p^{[\beta]}(s,a,m)$ has the interpolation property:

$$(1.3) \quad L_p^{[\beta]}(1 - k; a, m) = \begin{cases} m^{k-1} B_k\left(\frac{a}{m}\right) & (p \mid m); \\
\frac{m^{k-1}}{k} \left( B_k\left(\frac{a}{m}\right) - \frac{p^{k-1}}{k} B_k\left(\frac{(ap-1)}{m}\right)\right) & (p \nmid m)
\end{cases}$$

for all $k \geq 1$ with $k \equiv \beta \mod e$. Since $\mathbb{Z}_{>0,\alpha,\beta(\mod e)}$ is dense in the space $\beta + \frac{q}{p}\mathbb{Z}_p$ ($= \mathbb{Z}_p$ ($p > 2$), $2\mathbb{Z}_2$ or $1 + 2\mathbb{Z}_2$), the above interpolation property shows that $L_p^{[\beta]}(s,a,m)$ is determined independently of $\sigma$ (at least) on that space. In particular when $\beta \equiv 0 \mod e)$ and $p > 2$, $L_p^{[0]}(s,a,m) = -\zeta_p^{sh}(s;a,m)$ for $s \in \mathbb{Z}_p$. See also Appendix A for relations of $L_p^{[\beta]}(s,a,m)$ with Cohen’s Hurwitz zeta functions $\zeta_p(s,a)$.

In the present paper, we hope to make a small step towards the quest of J. Coates about existence of zeta functions on $\hat{\mathbb{Z}}$ with values in $\hat{\mathbb{Z}}$ ([W2], Introduction).

The mapping $\hat{\zeta}_{a,m}$ in Theorem 1.1 gives a 1-cocycle $G_{\mathbb{Q}(\mu_m)} \to \hat{\mathbb{Z}}(1)[[\hat{\mathbb{Z}}(-1)]]$ whose $(k-1)$st moment integral gives rise to a cohomology class in $H^1(G_{\mathbb{Q}(\mu_m)}, \hat{\mathbb{Z}}(k))$ for $k \geq 2$. In fact, we will show in Corollary 5.5:

$$\int_{\mathbb{Z}_p} b^{k-1} d\hat{\zeta}_{p,a,m}(\sigma)(b) = \frac{m^{k-1}}{k} B_k\left(\frac{a}{m}\right) \left(1 - \chi_p(\sigma)^k\right) \quad (\sigma \in G_{\mathbb{Q}(\mu_m)}, \ k \geq 2)$$

which implies that the $p$-adic image of the above cohomology class is torsion with order calculated explicitly by Bernoulli values. It is noteworthy that this cohomology class is closely related to the $\xi_{a,m}$-component of the $\mathbb{Z}(k)$-torsor “$P_{m,k} + (-1)^k eP_{m,k}$” over $\mu_m$ studied by P.Deligne in [De, Proposition 3.14, Lemme 18.5].
2. The Kummer-Heisenberg measure $\kappa_1$

2.1. Cyclic coverings. Let $F \subset \mathbb{C}$ be a finite extension of $\mathbb{Q}$ with the algebraic closure $\mathbb{C} \subset \mathbb{C}$. For any (normal) algebraic variety $V$ over $F$ and $F$-rational points $x, y \in V(F)$, we write $\pi^\text{et}(V; y, x)$ for the set of étale paths from $x$ to $y$ on the geometric variety $V \otimes \mathbb{F}$, and $\pi^\text{et}_1(V; x, x) = \pi^\text{et}(V; x, x)$ for the étale fundamental group with base point $x$. Denote by $\pi^\text{pro-p}_1(V, x)$ the maximal pro-$p$ quotient of $\pi^\text{et}_1(V, x)$, and by $\pi^\text{pro-p}_1(V; y, x)$ the natural push forward of $\pi^\text{et}_1(V; y, x)$ induced from the projection $\pi^\text{et}_1(V, x) \to \pi^\text{pro-p}_1(V, x)$.

For each $n \geq 1$, write $\mu_n := \exp(2\pi i/n)$ so that $\mu_n := \{1, \mu_n, \ldots, \mu_n^{n-1}\}$. Let

$$V_n := \mathbb{P}^1 \setminus \{0, \mu_n, \infty\},$$

where we understand $\{0, \mu_n, \infty\}$ is the abbreviation of $\{0, \infty\} \cup \mu_n$. Regard $V_n(\mathbb{C}) = \mathbb{C} \setminus \mu_n$. Let $\overline{01}_n$ be the tangential base point on $V_n$ represented by the unit tangent vector and denote for simplicity $\overline{01}$. Then, for each $n \geq 1$, there is a standard cyclic étale cover $p_n : V_n \to V_1$ given by $z \mapsto z^n$ which sends $\overline{01}_n$ to a Galois functor equivalent to $\overline{01}_1$ on $V_1$. Thus, without ambiguity, we may omit the index of $\overline{01}$ on $V_n$ and regard $(V_n, \overline{01})$ as a pointed étale cover over $(V_1, \overline{01})$. By standard Galois theory, it allows us to identify $\pi^\text{et}_1(V_n, \overline{01})$ as a subgroup of $\pi^\text{et}_1(V_1, \overline{01})$.

Let $x, y$ be the generators of $\pi^\text{et}_1(V_1, \overline{01})$ given by the loops based at $\overline{01}$ on $V_1 = \mathbb{P}^1 - \{0, 1, \infty\}$ running around 0, 1 once anti-clockwise respectively. Then, it is easy to see that, as a subgroup of it, $\pi^\text{et}_1(V_n, \overline{01})$ is freely generated by $x_n := x^n$ and $y_{b,n} := x^{-b}yx^b$ ($0 \leq b < n$).

2.2. Galois associators and Kummer-Heisenberg measure. Now, let $z$ be an $F$-point of $V_1 = \mathbb{P}^1 - \{0, 1, \infty\}$. We have the canonical comparison map

$$\pi(V_1(\mathbb{C}); z, \overline{01}) \to \pi^\text{et}_1(V_1; z, \overline{01})$$

from the set of homotopy classes of paths from $\overline{01}$ to $z$ on $V_1(\mathbb{C})$ to the étale paths from $\overline{01}$ to $z$ on $V_1 \otimes \mathbb{F}$. The Galois group $G_F$ acts on the profinite group $\pi^\text{et}_1(V_1, \overline{01})$ and its torsor of paths $\pi^\text{et}_1(V_1; z, \overline{01})$.

Let us fix an étale path $\gamma \in \pi^\text{et}_1(V_1(\mathbb{C}); z, \overline{01})$. For $\sigma \in G_F$, define the Galois associator for the path $\gamma$ by

$$f_\gamma(\sigma) := \gamma^{-1} \cdot \sigma(\gamma) \in \pi^\text{et}_1(V_1, \overline{01}),$$

where $\sigma(\gamma) := \sigma \circ \gamma \circ \sigma^{-1}$.

Write $\pi'$ for the commutator subgroup of a profinite group $\pi$. The abelianization of $f_\gamma(\sigma)$ is known (cf. [NW1], Prop. 1) to be expressed as:

$$f_\gamma(\sigma) \equiv x^{\rho_z,\gamma(\sigma)}y^{\rho_{1-z,\gamma}(\sigma)} \mod \pi^\text{et}_1(V_1, \overline{01})',$$

with the $\mathbb{Z}$-valued functions

$$\rho_{z,\gamma}, \rho_{1-z,\gamma} : G_F \to \mathbb{Z}(1)$$

the Kummer 1-cocycles associated with the roots of $z$ and $1 - z$. They are calculated along $\gamma$ with the base $(\xi_n)_{n \geq 1} \in \mathbb{Z}(1)$ respectively. For the latter $\rho_{1-z,\gamma}$, we understand the points $\overline{01}$ and $1 - z$ are connected by the unit segment $[0, 1]$ on $\mathbb{P}^1$ followed with the reversed path of $\gamma$ by $(\ast \mapsto 1 - \ast)$. We sometimes omit the mention to $\gamma$ when it is obvious from context.
Definition 2.1. Let $\sigma \in G_F$ and set

$$f_\gamma^0(\sigma) := x^{-\rho_\gamma(\sigma)}f_\gamma(\sigma) \quad (\sigma \in G_F).$$

which belongs to the subgroup $\pi_1(V_n, 0\overline{1}) \subset \pi_1(V_1, 0\overline{1})$ by (2.2) for every $n \geq 1$. Given $0 \leq b < n$, we define $\kappa_{z,\gamma}^{(n)}(\sigma)(b) \in \hat{\mathbb{Z}}$ by the congruence

$$f_\gamma^0(\sigma) \equiv \prod_{b=0}^{n-1} y_{b,n}^{\kappa_{z,\gamma}^{(n)}(\sigma)(b)}$$

modulo $\pi_1^0(V_n, 0\overline{1})$: the commutator subgroup of $\pi_1^0(V_n, 0\overline{1})$.

Proposition 2.2 (see [NW1] Lemma 1). For each $\sigma \in G_F$, the system of functions

$$\{\mathbb{Z}/n\mathbb{Z} \ni b \mapsto \kappa_{z,\gamma}^{(n)}(\sigma)(b) \in \hat{\mathbb{Z}}\}_{n \in \mathbb{N}}$$

running over $n \geq 1$ defines a $\hat{\mathbb{Z}}$-valued measure on $\hat{\mathbb{Z}}$. □

We shall denote the above measure by

$$\kappa_1(\gamma : 0\overline{1} \rightarrow z)(\sigma) \text{ or } \kappa_1(z,\gamma)(\sigma)$$

and call it the Kummer-Heisenberg measure associated with the path $\gamma : 0\overline{1} \rightarrow z$. We view it as an element of the Iwasawa algebra $\hat{\mathbb{Z}}[[\hat{\mathbb{Z}}]]$. Let $\hat{\mathbb{Z}}(1)$ denote the Galois module $\hat{\mathbb{Z}}$ acted on by $G_F$ by multiplication by the cyclotomic character, and $\hat{\mathbb{Z}}(-1)$ be its dual.

Proposition 2.3. The function

$$\kappa_1(\gamma : 0\overline{1} \rightarrow z) : G_F \rightarrow \hat{\mathbb{Z}}(1)[[\hat{\mathbb{Z}}(-1)]]$$

is a cocycle. Namely it holds that

$$\kappa_{z,\gamma}^{(n)}(\sigma\tau)(b) = \kappa_{z,\gamma}^{(n)}(\sigma)(b) + \chi(\sigma) \cdot \kappa_{z,\gamma}^{(n)}(\tau)(\chi(\tau)^{-1}b)$$

for $\sigma, \tau \in G_F$, $n \geq 1$, $b \in \mathbb{Z}/n\mathbb{Z}$.

Proof. By the definition of $f_\gamma$ (2.1), we have $f_\gamma(\sigma\tau) = f_\gamma(\sigma) \cdot f_\gamma(\tau)$, hence $f_\gamma^0(\sigma\tau) \equiv f_\gamma^0(\sigma) \cdot f_\gamma^0(\tau)$ modulo $\pi_1(V_n)'$. The assertion follows from this and the observation

$$\sigma(y_{b,n}) \equiv x^{\chi(\sigma)b}y^{\chi(\sigma)x^{-\chi(\sigma)b}} \equiv (y_{\chi(\sigma)b,n})^{\chi(\sigma)}$$

modulo $\pi_1(V_n)'$. □

Remark 2.4. In [NW1, Lemma 1], we introduced a compatible sequence $(\kappa_n)_n$ in the projective system $\lim_{\rightarrow} \hat{\mathbb{Z}}[\mathbb{Z}/n\mathbb{Z}]$ which forms a measure $\hat{\kappa} \in \hat{\mathbb{Z}}[[\hat{\mathbb{Z}}]]$. We call $\hat{\kappa}$ (resp. $\kappa_1$) the Kummer-Heisenberg measure in $e$-form (resp. $t$-form) in the terminology of Appendix B. These two measures are ‘oppositely directed’ mainly because of different choice of path conventions as follows. After identification $\hat{\mathbb{Z}} \sim \hat{\mathbb{Z}}(1)$ by $1 \mapsto (\xi_n = \exp(2\pi i/n))_n$, let $\epsilon$ denote the involution on $\hat{\mathbb{Z}}(1)$ induced by $\xi \mapsto \xi^{-1}$. Then, we have $\kappa_1(\sigma) = \epsilon \cdot \hat{\kappa}(\sigma)$ ($\sigma \in G_F$) as elements of $\hat{\mathbb{Z}}[[\hat{\mathbb{Z}}(1)]]$. 
3. Adelic Hurwitz measure

3.1. Paths to roots of unity. Fix $m \in \mathbb{N}_{>1}$ and let $a$ be an integer with $m \nmid a$. Let

\[ \tau : V_1 \rightarrow V_1 \]

be the involution given by $\tau(z) = z^{-1}$. For any path $\gamma$ on $V_1(\mathbb{C})$ from $0\Gamma$ to $\xi_m^a$, we set another path $\tilde{\gamma}$ from $0\Gamma$ to $\xi_m^{-a}$ by

\[ \tilde{\gamma} := \tau(\gamma) \cdot \Gamma_\infty \]

where $\Gamma_\infty$ is a path on $V_1(\mathbb{C})$ from $0\Gamma$ to $\infty\Gamma$ as on the Figure 1:

![Figure 1](image1)

Write $\frac{a}{m} = \lfloor \frac{a}{m} \rfloor + \{ \frac{a}{m} \}$ so that $0 \leq \{ \frac{a}{m} \} < 1$, and define the path $\Gamma_{a/m} : 0\Gamma \rightarrow \xi_m^a$ to be the composition $\Gamma_{\{a/m\}} \cdot x_{\{a/m\}}$, where $\Gamma_{\{a/m\}}$ is the path illustrated as in Figure 2.

![Figure 2](image2)

It is easy to see the following

**Lemma 3.1.** Along the above paths $\Gamma_{a/m} : 0\Gamma \rightarrow \xi_m^a$ and $\Gamma_{\{a/m\}} : 0\Gamma \rightarrow \xi_m^{-a}$, the associated Kummer 1-cocycles are coboundaries satisfying

\[ \rho_{\xi_m^a, \Gamma_{a/m}}(\sigma) = \frac{a}{m}(\chi(\sigma) - 1), \quad \rho_{\xi_m^{-a}, \Gamma_{\{a/m\}}}(\sigma) = -\frac{a}{m}(\chi(\sigma) - 1) \]

for $\sigma \in G_{\mathbb{Q}(\mu_m)}$.

**Proof.** The first formula is immediate from the definition and the identification $\hat{\mathbb{Z}} \cong \hat{\mathbb{Z}}(1)$ by $1 \mapsto (\xi_n)_{n \geq 1}$. For the second, it suffices to note that the image of $\Gamma_{a/m}$ by $\mathbb{P}^1 - \{0, 1, \infty\} \rightarrow \mathbb{P}^1 - \{0, \infty\}$ is topologically homotopic to the complex conjugate of $\Gamma_{a/m}$. \qed

**Remark 3.2.** It is worth noting that $\Gamma_{a}x^n = \Gamma_{a+n}$ for any $a \in \mathbb{Q}$ and $n \in \mathbb{Z}$. This additivity property does not hold for $\Gamma_{a}$ in general. Still, if $0 \leq a \leq 1$, then it holds that $\Gamma_{a} = \Gamma_{-a} = \Gamma_{n-a}x^{-n}$ for every $n \in \mathbb{Z}$. This last point will play a crucial role later in Lemma 5.7.
3.2. Translation of a measure. Let \( p_n : V_n \to V_1 \) be the cyclic cover of degree \( n \) considered in \S 2.1. For an étale path \( \gamma : 01 \to z \) on \( V_1 \), we shall write

\[
\gamma_n(= (\gamma)_n) : \overrightarrow{01}_n \to z^{1/n}
\]

to denote the lift of \( \gamma \) to \( V_n \) for all \( n \geq 1 \). Let us fix \( \sigma \in G_F \). Note that the end point \( z^{1/n} \) may or may not be fixed by the \( \sigma \). In fact, we can interpret the fact

\[
f^\gamma_0(\sigma) = (\gamma \cdot x^{p_\gamma(\sigma)})^{-1}\sigma(\gamma) \in \pi^1(\overrightarrow{V_1}, \overrightarrow{01})' \subset \pi_1(V_n, \overrightarrow{01})
\]

from (2.2) so that the lifts of \( \gamma \cdot x^{p_\gamma(\sigma)} \) and of \( \sigma(\gamma) \) departing at \( \overrightarrow{01}_n \) on \( V_n \) and at the same point \( \sigma(z^{1/n}) = \xi^{p_\gamma(\sigma)}z^{1/n} \). Since the lift \( (x^{p_\gamma(\sigma)})_n \) of \( x^{p_\gamma(\sigma)} \) from \( \overrightarrow{01}_n \) ends at \( \xi^{p_\gamma(\sigma)}\overrightarrow{01}_n \), the subsequent path \( \gamma_n, \sigma \) should lift \( \gamma \) so as to start from that point \( \xi^{p_\gamma(\sigma)}\overrightarrow{01}_n \) with ending at the point \( \sigma(z^{1/n}) \) on \( V_n \):

\[
(\sigma(\gamma))_n - \overrightarrow{\xi^{p_\gamma(\sigma)}\overrightarrow{01}_n} \to \gamma_n, \sigma - \gamma(s) = \sigma(z^{1/n}).
\]

In summary, writing \( (\sigma(\gamma))_n \) for the lift of \( \sigma(\gamma) \) from \( \overrightarrow{01}_n \) on \( V_n \), we may express \( f^\gamma_0(\sigma) \) as the composition of those three paths

\[
f^\gamma_0(\sigma) = (x^{p_\gamma(\sigma)})_n^{-1} \cdot \gamma_n, \sigma^{-1} \cdot (\sigma(\gamma))_n
\]

on \( V_n \).

Below, we shall see magnification of the base space \( \hat{\mathbb{Z}} \) on a coset \( s + r\hat{\mathbb{Z}} \) \((s, r \in \mathbb{Z}, r \geq 1)\) under the measure \( \kappa_1(z)_\gamma(\sigma) \) can be interpreted as a twisted lifting of the reference path \( \gamma : \overrightarrow{01} \to z \) to \( V_{\gamma} \) followed with ‘s-rotated’ embedding by \( V_{\gamma} \to V_1 \).

Set an ‘s-modified’ path \( \gamma_{(-s)} : \overrightarrow{01} \to z \), for the given path \( \gamma : \overrightarrow{01} \to z \) on \( V_1 \), by

\[
\gamma_{(-s)} := \gamma \cdot x^{-s}.
\]

It follows easily that

\[
\rho_{z, \gamma_{(-s)}}(\sigma) = \rho_{z, \gamma}(\sigma) - s(\chi(\sigma) - 1) \quad (\sigma \in G_F).
\]

Suppose that \( \xi_r, z^{1/r} \in F \). Write

\[
\gamma_r = (\gamma)_r : \overrightarrow{01}_r \to z^{1/r},
\]

\[
\gamma_{(-s), r} = (\gamma_{(-s)})_r : \overrightarrow{01}_r \to \xi_r^{-s}z^{1/r}
\]

for the lifts of the paths \( \gamma \) and \( \gamma_{(-s)} \) by \( p_r : V_r \to V_1 \) respectively, and

\[
\gamma_{(-s), r} = p_r(\gamma_{(-s)})_r : \overrightarrow{01}_r \to \xi_r^{-s}z^{1/r}
\]

for the image of \( \gamma_{(-s), r} \) by the immersion \( j_r : (V_r, \overrightarrow{01}_r) \to (V_1, \overrightarrow{01}) \). It follows that

\[
\rho_{z^{1/r}, \gamma_{(-s), r}}(\sigma) = \rho_{z^{1/r}, \gamma_{(-s)}}(\sigma) - \frac{s}{r}(\chi(\sigma) - 1) \quad (\sigma \in G_F).
\]

Lemma 3.3. Notations being as above, with assumptions \( \xi_r, z^{1/r} \in F \) and \( \sigma \in G_F \).

(i) For every \( n \geq 1 \), it holds that

\[
\kappa^{(nr)}_{z, \gamma}(vr + s\chi(\sigma)) = \kappa^{(n)}_{\xi_r^{vr}, z^{1/r}, \gamma_{(-s), r}}(\sigma)(v) \quad (v = 0, \ldots, n - 1),
\]

where \( vr + s\chi(\sigma) \) in LHS is regarded \( \in (\mathbb{Z}/nr\mathbb{Z}) \).
(ii) For any continuous function \( \varphi \) on \( \hat{\mathbb{Z}} \), we have
\[
\int_{s\chi(\sigma)+r\mathbb{Z}} \varphi(b) d\chi_1(\gamma : \hat{\mathbb{O}} \to z)(\sigma)(b) = \int_{\hat{\mathbb{Z}}} \varphi(rv + s\chi(\sigma)) d\chi_1(\gamma_{(-s),r};\xi_r^{-s}z^{1/r})(\sigma)(v).
\]

**Proof.** In this proof, for \( n \geq 1 \), we denote \( \pi(n) := \pi_{\hat{\mathbb{O}}}(V_n, \hat{\mathbb{O}}) \) and write \( \varpi_{nr} : \pi(nr) \to \pi(n) \) for the surjection induced from the open immersion \((V_{nr}, \hat{\mathbb{O}}_{nr}) \to (V_n, \hat{\mathbb{O}}_n)\). Note that, among the standard generators \( x^{nr}, y_{b,nr} \) \((b = 0, \ldots, nr-1)\) of \( \pi(nr) \), only \( x^{nr} \) and \( y_{v,nr} \) \((v = 0, \ldots, n-1)\) survive via \( \varpi_{nr} \) to be \( x^n, y_{v,n} \) \((v = 0, \ldots, n-1)\) in \( \pi(n) \).

Noting that \( x^{-u}yx^n = y_{u,nr} \equiv y_{u+nrk,nr} \mod \pi(nr)' \) for \( u, k \in \mathbb{Z} \), we see from Definition 2.1 that
\[
\int_{s\chi(\sigma)+r\mathbb{Z}} \varphi(b) d\chi_1(\gamma : \hat{\mathbb{O}} \to z)(\sigma)(b) = \int_{\mathbb{Z}} \varphi(rv + s\chi(\sigma)) d\chi_1(\gamma_{(-s),r};\xi_r^{-s}z^{1/r})(\sigma)(v).
\]

which should map via \( \varpi_{nr} \) to the product over those \( v \) multiples of \( r \):
\[
\varpi_{nr} \left( x^{nr} \cdot f^y_\gamma(\sigma) \cdot x^{-s\chi(\sigma)} \right) = \prod_{v=0}^{n-1} y_{v,nr}^{\xi_r^{-sr}(\sigma)(v+s\chi(\sigma))} \mod \pi(n)'
\]

as \( \pi(n)' \supset \varpi_{nr}(\pi(nr)' \pi(n)') \). We shall interpret the LHS of the above expression (3.5) by applying the composition diagram (3.1) to the path \( \gamma_{(-s)} : \hat{\mathbb{O}} \to z \) (3.2) on \( V_1 \) and its lift \( (\gamma_{(-s)})_r = \gamma_{(-s),r} \) on \( V_r \):

We first derive:
\[
x^{s\chi(\sigma)} \cdot f^y_\gamma(\sigma) \cdot x^{-s\chi(\sigma)} = x^{s\chi(\sigma)} \cdot x^{-\rho_{z,s}(\gamma)} \cdot (\gamma \cdot x^{-s})^{-1} \sigma(\gamma \cdot x^{-s}) = (\gamma_{(-s)} \cdot x^{\rho_{z,s}(\gamma)} \cdot (\gamma \cdot x^{-s})^{-1} \sigma(\gamma \cdot x^{-s})).
\]

By (3.3), the former factor of path composition reads on \( V_r \)
\[
(\gamma_{(-s)})_r \cdot \left( x^{\rho_{z,s}(\gamma)}(\sigma) \right) = (\gamma \cdot x^{-s}) \cdot \left( x^{\rho_{z,s}(\gamma) - s(\chi(\sigma) - 1)} \right).
\]

where \( (\gamma_{(-s)})_r \cdot \sigma \) stands for a suitable lift of \( \gamma \) on \( V_r \), which arrives at the same end point on \( V_r \) as the latter \( \sigma \)-transformed factor
\[
(\sigma(\gamma_{(-s)})) : \hat{\mathbb{O}}_r \to \sigma(\xi_r^{-s}z^{1/r}).
\]

It turns out that \( (\gamma_{(-s)})_r \cdot \sigma \) starts at \( \xi_r^{\rho_{z,s}(\gamma) - s(\chi(\sigma) - 1)} \cdot \hat{\mathbb{O}}_r \), which is equal to \( \hat{\mathbb{O}}_r \) by our assumption \( \xi_r, z^{1/r} \in F \). Thus we conclude
\[
(\gamma_{(-s)})_r = (\gamma_{(-s),r} \cdot (\gamma \cdot x^{-s})_r).
\]
By virtue of this and (3.4), applying \( \varpi_r \) to (3.7) implies
\[
\varpi_r \left( x^{-\rho_s} \gamma_{(-s)}(\sigma) \cdot (\gamma_{(-s)})_{r,\sigma}^{-1} \cdot \sigma(\gamma_{(-s),r}) \right) = \varpi_r \left( (x_r)^{-\rho_s} \gamma_{(-s),r}^{-1} \cdot \sigma(\gamma_{(-s),r}) \right)
\]
\[
= x^{-\rho_s} \gamma_{(-s),r}^{-1} \cdot \gamma_{(-s),r,s} \cdot \sigma(\gamma_{(-s),r,s})
\]
\[
= f_{\gamma_{(-s),r}}(\sigma)
\]
\[
= \prod_{e=0}^{n-1} y_{e,m} \left( z e_s^{-1} \gamma_{(-s),r,s} (\gamma_{(-s),s})^{(v)} \right) \mod \pi(n)'.
\]

This, combined with (3.6)-(3.7) and the compatibility \( \varpi_{nr} = \varpi_r |_{\varpi(nr)} \), proves (i). The assertion (ii) is just a formal consequence of (i). \( \square \)

Suppose we are given a measure \( \mu \in \hat{\mathbb{Z}}[[\hat{\mathbb{Z}}]] \). Let \( m, a \in \mathbb{Z} \) be integers as in §3.1 and pick \( n \in \hat{\mathbb{Z}}^\times \). Consider the coset \( R_{av,m} := \frac{am}{m} + \hat{\mathbb{Z}} \) of \( \hat{\mathbb{Z}} \) in \( \mathbb{Q}_f = \hat{\mathbb{Z}} \otimes \mathbb{Q} \). Then, obviously,
\[
R_{av,m} := m \cdot Q_{av,m} = \mathcal{O} + m\mathbb{Z} \subset \hat{\mathbb{Z}}.
\]

We define the measure \([m, av]_\mu(\mu)\) on \( R_{av,m} \) by assigning to each open subset \( U \subset R_{av,m} \) the value \( \mu(U) \), where \( U' \) is the inverse image of \( U \) by the affine map \( t \mapsto mt + av \) (\( t \in \hat{\mathbb{Z}} \)). Note that Lemma 3.3 (ii) reads:
\[
(3.9) \quad \varphi_1(\gamma : 0 \mathcal{O} \to z)(\sigma)|_{s\chi(\sigma) + m\mathbb{Z}} = [m, s\chi(\sigma)]_s \left( \varphi_1(\gamma_{(-s),m\mathbb{Z}} : \mathcal{O} \to \xi^{-s}z^{1/m})(\sigma) \right)
\]
for \( \sigma \in \mathcal{G}_\mathbb{F} \), \( m, s \in \mathbb{Z} \), \( m \geq 1 \), where \( *|_{s\chi(\sigma) + m\mathbb{Z}} \) in \( \text{LHS} \) designates the restricted measure on \( R_{s\chi(\sigma),m} = s\chi(\sigma) + m\mathbb{Z} \subset \hat{\mathbb{Z}} \).

Let \( \iota \) denote the action of the complex conjugation on \( \hat{\mathbb{Z}}(1)[[\hat{\mathbb{Z}}(-1)]] \), i.e., the action of \(-1 \in \hat{\mathbb{Z}}^\times \). It is straightforward to see
\[
(3.10) \quad \iota \circ [m, av]_* = [m, -av]_* \circ \iota.
\]

Now we are ready to introduce the fundamental object of our study. Let \( m > 1 \) and \( a \in \mathbb{Z} \) as above, and let \( \Gamma_{a/m} \in \pi(V_1(\mathbb{C}); \xi_{a/m}^{-} \mathcal{O}) \) be the path introduced in §3.1.

**Definition 3.4 (\( \hat{\mathbb{Z}} \)-Hurwitz and adelic Hurwitz measure).** For each \( \sigma \in G_{\mathbb{Q}(\mu_m)} \) we define the \( \hat{\mathbb{Z}} \)-Hurwitz measure \( \zeta_{a/m}(\sigma) \in \hat{\mathbb{Z}}[[\hat{\mathbb{Z}}]] \) and the adelic Hurwitz measure \( \zeta_{a,m}(\sigma) \in \hat{\mathbb{Z}}[[\hat{\mathbb{Z}}]] \) by the formulas
\[
\zeta_{a/m}(\sigma) := \varphi_1(\mathcal{O} \to \xi^{-a})(\sigma) + \iota \left( \varphi_1(\mathcal{O} \to \xi^a)(\sigma) \right);
\]
\[
\zeta_{a,m}(\sigma) := [m, a\chi(\sigma)]_* \zeta_{a/m}(\sigma).
\]

4. Geometrical interpretation of translation of measure

In this section we address the fact that translation of Kummer-Heisenberg measure by \([m, ax]_* \) corresponds to path composition with the loop \( x^a \). We work however only with \( p \)-adic measures.

4.1. \( \ell \)-adic Galois polylogarithms. Let \( \gamma \) be an étale path on \( V_1 \) from \( \mathcal{O} \) to an \( F \)-rational (possibly tangential) point \( z \). Let \( \mathbb{Q}_p\langle X, Y \rangle \) be the non-commutative power series ring in two variables \( X, Y \), and write \( \mathcal{E}^\prime : \pi_1^{\pro-p}(V_1, \mathcal{O}) \to \mathbb{Q}_p\langle X, Y \rangle \) for the embedding that sends the standard generators \( x, y \) to \( \exp(X), \exp(Y) \). We define \( I_Y \) to be the ideal of \( \mathbb{Q}_p\langle X, Y \rangle \) generated by monomials containing \( Y \) twice or more.
For $\sigma \in G_F$, set
\[
\Lambda_{\gamma}(\sigma) := \mathcal{E}(f_{\gamma}(\sigma));
\]
\[
\Lambda_{\gamma}(\sigma) := \mathcal{E}(f^*_{\gamma}(\sigma)) = \exp(-\rho_{z,\gamma}(\sigma)X) \cdot \Lambda_{\gamma}(\sigma).
\]

**Definition 4.1.** Define $\ell$-adic polylogarithms $\ell_{ik}(z,\gamma), \text{Li}_k(z,\gamma) : G_F \to \mathbb{Q}_\ell$ by the congruence expansion
\[
\log \Lambda_{\gamma}(\sigma) \equiv \rho_{z,\gamma}(\sigma) + \sum_{k=1}^{\infty} (-1)^{k-1} \ell_{ik}(z,\gamma) (\text{ad}X)^{k-1}(Y),
\]
\[
\log \Lambda_{\gamma}(\sigma) \equiv \sum_{k=1}^{\infty} (-1)^{k-1} \text{Li}_k(z,\gamma) (\text{ad}X)^{k-1}(Y)
\]
modulo the ideal $I_Y$.

**Proposition 4.2.** The family of functions $\{\rho_{z,\gamma}, \ell_{ik}(z,\gamma), \text{Li}_k(z,\gamma) : G_F \to \mathbb{Q}_p\}_{k \geq 1}$ satisfy (i)
\[
\text{Li}_k(z,\gamma) = \sum_{i=1}^{k} \frac{\rho_{z,\gamma}^{-i}}{(k+1-i)!} \ell_i(z,\gamma),
\]
(ii)
\[
\ell_{ik}(z,\gamma) = \sum_{s=0}^{k-1} \frac{B_s}{s!} \rho_{z,\gamma}^{s-1} \text{Li}_{k-s}(z,\gamma)
\]
for $k = 1, 2, \ldots$.

In fact, this proposition is a formal consequence of the following lemma:

**Lemma 4.3.** Let $K$ be a field of characteristic zero, and suppose that two sequences $\{b_i\}_{i \geq 0}, \{u_i\}_{i \geq 0}$ in $K$ satisfy the congruence
\[
e^{-u_0X} e^{u_0X + \sum_{s=0}^{\infty} u_{k+1}(\text{ad}X)^k(Y)} \equiv e^{b_0X + \sum_{s=0}^{\infty} b_{k+1}(\text{ad}X)^k(Y)} \mod I_Y
\]
as non-commutative power series in $X, Y$. Then, $b_0 = 0$ and, for $k = 1, 2, \ldots$,
\[
b_k = \sum_{i=1}^{k} \frac{(-u_0)^{k-i}}{(k+1-i)!} u_i,
\]
\[
u_k = \sum_{s=0}^{k-1} \frac{B_s}{s!} (-u_0)^s b_{k-s},
\]
where $B_0, B_1, \ldots$ are Bernoulli numbers defined by $\sum_{s=0}^{\infty} \frac{B_s}{s!} T^s = \frac{T}{e^T-1}$.

**Proof.** We make use of the classical Campbell-Hausdorff formula
\[
\log(e^\alpha e^\beta) \equiv \beta + \sum_{n=0}^{\infty} \frac{B_n}{n!} (\text{ad}\alpha)^n(\alpha) \mod \text{deg}(\alpha) \geq 2.
\]
Set $-\alpha = \sum_{i=0}^{\infty} b_{i+1}(\text{ad}X)^i(Y)$ and $-\beta = u_0X$ so that congruences mod $\text{deg}(\alpha) \geq 2$ derive those mod $I_Y$. It follows that $\log(e^{u_0X} e^{b_1Y+b_2[X,Y]+\ldots})$ is congruent to $u_0X + \sum_{s=0}^{\infty} (\sum_{s=0}^{k} \frac{B_s}{s!} (-u_0)^s b_{k+1-s}) (\text{ad}X)^k(Y) \mod I_Y$. This is equivalent to the equality
\[
\sum_{k=0}^{\infty} u_{k+1} T^k = \left(\frac{-u_0T}{e^{-u_0T} - 1}\right) \sum_{k=0}^{\infty} b_{k+1} T^k.
\]
The assertion follows from this immediately. $\square$
4.2. **Extension for $\mathbb{Q}_p$-paths.** Let $\pi_{\mathbb{Q}_p}(\overline{\mathbb{O}})$ be the pro-algebraic hull of the image of the above embedding $\phi' : \pi_1^{\text{pro-p}}(V_1, \overline{\mathbb{O}}) \to \mathbb{Q}_p(X, Y)$, and extend it to the inclusion of path torsors $\pi_1^{\text{pro-p}}(V_1; z, \overline{\mathbb{O}}) \to \pi_{\mathbb{Q}_p}(z, \overline{\mathbb{O}})$ naturally. The elements of $\pi_{\mathbb{Q}_p}(\overline{\mathbb{O}})$, $\pi_{\mathbb{Q}_p}(z, \overline{\mathbb{O}})$ will be simply called $\mathbb{Q}_p$-paths, and the action of the Galois group $G_F$ on the pro-$p$ paths extends to that on the $\mathbb{Q}_p$-paths in the obvious manner.

For each $\mathbb{Q}_p$-path $\gamma : \overline{\mathbb{O}} \to z$ and $\sigma \in G_F$, we may define the Galois associator $f_{\gamma}(\sigma) := \gamma^{-1} \cdot (\sigma(\gamma)) \in \pi_{\mathbb{Q}_p}(\overline{\mathbb{O}})$ extending (2.1). Then, define $\rho_{z, \gamma}, li_k(z, \gamma) : G_F \to \mathbb{Q}_p (k = 1, 2, \ldots)$ as the coefficients in $\log(f_{\gamma}(\sigma))$ so as to extend the congruence in Definition 4.1 mod $I_Y$, and then, define $\text{Li}_k(z, \gamma) : G_F \to \mathbb{Q}_p (k = 1, 2, \ldots)$ as the coefficients of $\log((\exp(-\rho_{z, \gamma}(\sigma))X \cdot f_{\gamma}(\sigma))$ again as the extension of Definition 4.1. Then, it is simple to see that the identities in Proposition 4.2 hold true for $\mathbb{Q}_p$-paths $\gamma : \overline{\mathbb{O}} \to z$ in the same forms.

4.3. **Relation with $\chi_{1,p}$.** We now arrive at the stage to connect the $\ell$-adic polylogarithms $\text{Li}_k$ and the Kummer-Heisenberg measure $\chi_1$. In [NW1], we showed that, for pro-$p$ paths $\gamma : \overline{\mathbb{O}} \to z$, the function $\text{Li}_k(z, \gamma)$ multiplied by $(k - 1)!$ can be written by a certain polylogarithmic character $\chi_k(z, \gamma) : G_F \to \mathbb{Z}_p$ defined by Galois transformations of certain sequence of numbers of forms $\prod_{n=0}^{n-1}(1 - \xi z^1/p^n)^{k-1/p^n} (\xi \in \mu_{p^n}, n \geq 1)$. This enabled us to express $\text{Li}_k(z, \gamma)(\sigma) (\sigma \in G_F)$ by the moment integral $\int_{\mathbb{Z}_p} b^{k-1} d\chi_{1,p}(\sigma)(b)$ over the $p$-adic measure $\chi_{1,p}(\sigma)$ which is by definition the image of the Kummer-Heisenberg measure $\chi_{1}(\sigma)$ (§2.2) by the projection $\hat{\mathbb{Z}}[[\hat{\mathbb{Z}}]] \to \mathbb{Z}_p[[\mathbb{Z}_p]]$.

A generalization of this phenomenon has been investigated in [NW3] for some more general $\mathbb{Q}_p$-paths of the form $\gamma x^{a/m}$. We summarize the result as follows:

**Proposition 4.4** ([NW3] (7.3)). Let $\gamma : \overline{\mathbb{O}} \to z$ be a pro-$p$ path. Then, for any $\alpha \in \mathbb{Q}_p$, we have

$$\text{Li}_k(z, \gamma x^a)(\sigma) = \frac{1}{(k-1)!} \int_{\mathbb{Z}_p} (b + \alpha \chi(\sigma))^{k-1} d\chi_{1,p}(\overline{\mathbb{O}} \to z)(b) \quad (\sigma \in G_F).$$

**Proof.** We just translate [NW3] (7.3) from $e$-form to $t$-form in the terminology of Appendix. In $e$-form, it reads (with $\delta := \gamma$, $\alpha := -\frac{\xi}{a}, \hat{k}_p := \chi_{z, \gamma}$ in loc.cit.):

$$\hat{\chi}_k^{x, \delta}(\sigma) = \int_{\mathbb{Z}_p} (b - a \chi(\sigma))^{k-1} d\hat{k}_p(\sigma)(b).$$

Let $\gamma x^a$ be the $t$-path reciprocally corresponding to the $e$-path $x^{a, \delta}$. In RHS, we regard the measure $\hat{k}_p$ as the $p$-adic image of $\hat{k}(\delta)$ of Remark 2.4 which can be switched into the $e$-form $\chi_{1,p}(\gamma)$ to obtain

$$\int_{\mathbb{Z}_p} (b - a \chi(\sigma))^{k-1} d\hat{k}_p(\sigma)(b) = \int_{\mathbb{Z}_p} (-b - a \chi(\sigma))^{k-1} d\chi_{1,p}(\sigma)(b).$$

At the same time, we may convert the LHS to $t$-form by Appendix (B.11) and (B.13) as:

$$\hat{\chi}_k^{x, \delta}(\sigma) = -(k-1)! \cdot \mathcal{L}_{ik}(x^a \delta : \overline{\mathbb{O}} \to z)(\sigma) = (-1)^{k-1}(k-1)! \text{Li}_k(\gamma x^a : \overline{\mathbb{O}} \to z)(\sigma).$$

The formula of the proposition follows from combination of these identities. □
5. Consequence of Inversion Formula

5.1. Pro-p inversion formula. We start this section with the main technical result.

Let $a, m$ be integers with $m > 1$, $m 
mid a$, and fix the $m$-th root of unity $z := \xi_m^a \in \mu_m$ and set $F = \mathbb{Q}(z)$. Pick any path $\gamma : \overline{0}1 \rightarrow z$ in $\pi_1^{\text{pro-$p$}}(\mathbb{P}^1 - \{0, 1, \infty\}, z, \overline{0})$ and let $\tilde{\gamma} : \overline{0}1 \rightarrow z^{-1}$ be the associated path defined in §3.1.

By the assumption $z \in \mu_m$, using the $p$-adic cyclotomic character $\chi_p : G_F \rightarrow \mathbb{Z}_p$, we may suppose that the Kummer 1-cocycle $\rho_{z, \gamma} : G_F \rightarrow \mathbb{Z}_p^\times$ is of a 1-coboundary form

$$\rho_z(\sigma) = \alpha(\chi_p(\sigma) - 1) \quad (\sigma \in G_F)$$

for a unique constant $\alpha \in \frac{a}{m} + \mathbb{Z}_p$. Since we do not assume $p \nmid m$, the constant $\alpha \in \mathbb{Q}_p$ may generally have denominator, while $\rho_z(\sigma) \in \mathbb{Z}_p$.

Theorem 5.1. Notations being as above, we have

$$\text{Li}_k(\xi_m^a)_{\gamma x^\alpha}(\sigma) + (-1)^k \text{Li}_k(\xi_m^a)_{\gamma x^{-\alpha}}(\sigma) = \frac{1}{k!} B_k(\alpha)(1 - \chi_p(\sigma)^k).$$

for $\sigma \in G_F$ and $k \geq 1$.

This result generalizes [W3, Theorem 10.2], where only the case $p \nmid m$ was considered. Here, we shall present a proof using the inversion formula for $p$-adic Galois polylogarithms [NW2]. For $\sigma \in G_F$, consider the $\ell$-adic polylogarithmic characters (for $\ell = p$) $\hat{\chi}_k(z)_{\gamma}(\sigma)$, $\hat{\chi}_k(\overline{z})_{\gamma}(\sigma)$ along those pro-$p$ paths $\gamma$ and $\tilde{\gamma}$. In [NW2, 6.3], we showed an inversion formula for $\gamma$ and $\tilde{\gamma}$ in the following form*

$$\hat{\chi}_k(z)_{\gamma}(\sigma) + (-1)^k \hat{\chi}_k(\overline{z})_{\gamma}(\sigma) = -\frac{1}{k!} \left\{ B_k(-\rho_z(\sigma)) - B_k \cdot \chi_p(\sigma)^k \right\} \quad (\sigma \in G_F),$$

where $B_k(T)$ is the Bernoulli polynomial defined by $\sum_{k=0}^{\infty} B_k(T) \frac{u^k}{k!} = \frac{ue^{uw}}{e^w - 1}$ and $B_k = B_k(0)$. Apply to (5.2) the translation formula

$$\text{Li}_k(z)_{\gamma}(\sigma) = \frac{(-1)^{k-1}}{(k-1)!} \hat{\chi}_k(z)_{\gamma}(\sigma) \quad (\sigma \in G_F, k \geq 1)$$

for which we refer the reader to §5 Appendix (B.13) and (B.14), and obtain

$$\text{Li}_k(\frac{1}{z})_{\gamma}(\sigma) + (-1)^k \text{Li}_k(z)_{\gamma}(\sigma) = \frac{1}{k!} \left\{ B_k(-\rho_z(\sigma)) - B_k \cdot \chi_p(\sigma)^k \right\} \quad (\sigma \in G_F).$$

Observe that this formula already gives a special case of Theorem 5.1 where $\alpha = 0$ and $\rho_z(\sigma) = 0$. What we shall do from now is to deform this formula into a form involved with the $\mathbb{Q}_p$-paths $\gamma x^{-\alpha}$ and $\tilde{\gamma} x^\alpha$. In fact, it follows from Proposition 4.4, we generally have

$$\text{Li}_k(z)_{\gamma x^{-\alpha}}(\sigma) = \frac{1}{(k-1)!} \sum_{i=0}^{k-1} \binom{k-1}{i} (\alpha \chi_p(\sigma))^i \text{Li}_{k-i}(z)_{\gamma}(\sigma),$$

hence the LHS of Theorem 5.1 can be written as:

$$\text{Li}_k(\frac{1}{z})_{\gamma x^{-\alpha}}(\sigma) + (-1)^k \text{Li}_k(z)_{\gamma x^{-\alpha}}(\sigma)$$

$$= \frac{1}{(k-1)!} \sum_{i=0}^{k-1} \binom{k-1}{i} (\alpha \chi_p(\sigma))^i \left( \text{Li}_{k-i}(\frac{1}{z})_{\gamma}(\sigma) + (-1)^{k-i} \text{Li}_{k-i}(z)_{\gamma}(\sigma) \right).$$

*See Appendix (B.14). The path $\tilde{\gamma}$ from $\overline{0}1$ to $z^{-1} \in \mu_m$ in $\ell$-form here reciprocally corresponds to the path $(0, 1][\xi_m^a](1, \infty) \cdot f_2(\gamma)$ in $\ell$-form with the notation of [NW2] §6.3.
To complete the proof of Theorem 5.1, by comparing (5.4) and (5.5), we are now reduced to the following core lemma:

**Lemma 5.2.** Let \( k \) be a positive integer, and set \( J_s := -\frac{1}{2}\{B_s(-\rho_z(\sigma)) - B_s \cdot \chi_p(\sigma)^s\} \) for \( s = 1, \ldots, k \) and \( \sigma \in \mathbb{G}_F \). Then, we have

\[
\frac{1}{k} B_k(\alpha) (\chi_p(\sigma)^k - 1) = \sum_{i=0}^{k-1} \binom{k-1}{i} \alpha^i \chi_p(\sigma)^i J_{k-i}(\sigma).
\]

**Proof.** For simplicity, we omit \( \sigma \) in this proof. Noting \( \frac{1}{k-1} \binom{k-1}{i} = \frac{1}{k} \binom{k}{i} \), the RHS of the lemma can be computed as

\[
\text{RHS} = \left[ -\sum_{i=0}^{k-1} \frac{1}{k} \binom{k}{i} \alpha^i B_{k-i}(-\rho_z) \right] + \left[ \frac{\chi_p}{k} \sum_{i=0}^{k-1} \binom{k}{i} \alpha^i B_{k-i} \right]
\]

To simplify the above former term, we make use of the Bernoulli addition formula \( B_k(y + x) = \sum_{s=0}^{k} \binom{k}{s} B_s(y) x^{k-s} \) to expand \( B_m(-\rho_z) \) as follows:

\[
B_k(-\rho_z) = \sum_{s=0}^{k} \binom{k}{s} B_s(\alpha)(-\alpha)^{k-s} \chi_p^{k-s}.
\]

Using (5.7) we then compute

\[
\left[ -\sum_{i=0}^{k-1} \frac{1}{k} \binom{k}{i} \alpha^i B_{k-i}(-\rho_z) \right] = -\sum_{i=0}^{k-1} \sum_{s=0}^{k-i} \frac{1}{k} \binom{k}{i} \binom{k-i}{s} (-1)^{k-i-s} \alpha^{k-s} \chi_p^{k-s} B_s(\alpha)
\]

\[
= -\sum_{s=1}^{k} \sum_{i=0}^{k-s} F(i, s) - \sum_{i=0}^{k-1} F(i, 0)
\]

\[
= -\frac{1}{k} \sum_{s=1}^{k} \alpha^{k-s} \chi_p^{k-s} B_s(\alpha) \left[ \sum_{i=0}^{k-s} \binom{k}{i} \binom{k-i}{s} (-1)^{k-i-s} \right] - \frac{1}{k} \alpha^k \chi_p \left[ \sum_{i=0}^{k-1} \binom{k}{i} (-1)^{k-i} \right]
\]

\[
= -\frac{1}{k} B_k(\alpha) + \frac{1}{k} \alpha^k \chi_p.
\]

Putting this into (5.6), we settle the proof of the lemma. \( \square \)

Thus, our proof of Theorem 5.1 is completed. \( \square \)

**Remark 5.3.** We mention that the proof in [W3] also carries over in the case \( p \mid m \); One needs only consider rational paths in \( \pi^{\text{prop}}(V_1; \xi_m, \overline{01}) \otimes \mathbb{Q} \) and in \( \pi^{\text{prop}}(V_1, \overline{01}) \otimes \mathbb{Q} \). The embedding of the latter \( \pi_1 \otimes \mathbb{Q} \) into \( \mathbb{Q}_p \langle X, Y \rangle \) extends to that of the former pro-\( p \) path space into \( \mathbb{Q}_p \langle X, Y \rangle \).
Remark 5.4. If \( p \nmid m \), then for any \( \alpha \in \mathbb{Z}_p \) there is \( \gamma \in \pi^{pro-p}(V_1; \xi_m^a, 0\overline{1}) \) such that \( \rho_{z, \gamma} = \alpha(\chi_p - 1) \). Hence we have
\[
\frac{(k-1)!}{\chi_p(\sigma)^k - 1} \left( \text{Li}_k(\xi_m^a)^{\gamma x^a}(\sigma) + (-1)^k \text{Li}_k(\xi_m^a)^{\gamma x - a}(\sigma) \right) = -\frac{B_k(\alpha)}{k}
\]
as long as \( \chi_p(\sigma)^k \neq 1 \). A key observation here is the following: Taking \( \alpha = 0 \) we get values of the Riemann zeta function at negative integers (cf. [W2]), while taking \( \alpha = \frac{a}{m} \in \mathbb{Q}^x \) we get values of Hurwitz zeta function \( \zeta(s, \frac{a}{m}) \) at negative integers. If we choose \( \gamma \) from topological paths \( \Gamma_{a/m} \in \pi(V_1(\mathbb{C}); \xi_m^a, 0\overline{1}) \) (§3.1), then we get \( -\frac{1}{k}B_k(\frac{a}{m}) \) for every choice of rational prime \( p \).

5.2. Moment integrals of \( p \)-adic Hurwitz measure. First we shall rewrite the formula in Theorem 5.1 in terms of measures \( \chi_{1,p}(0\overline{1}^{-\gamma} \xi_m^a) \) and \( \chi_{1,p}(0\overline{1}^{-\gamma} \xi_m^a) \) after multiplied by \( m^{k-1} \). Set \( \alpha := \frac{a}{m} \in \mathbb{Q} \). By Proposition 4.4 we find that, for \( \sigma \in G_F \),
\[
(5.8) \quad m^{k-1}\text{Li}_k(\xi_m^a)^{\gamma x^a}(\sigma) = \frac{m^{k-1}}{(k-1)!} \int_{\mathbb{Z}_p} (v + \alpha \chi_p(\sigma))^{k-1} d\left( \chi_{1,p}(0\overline{1}^{-\gamma} \xi_m^a)(\sigma) \right)(v)
\]
\[
= \frac{1}{(k-1)!} \int_{\mathbb{Z}_p} b^{k-1} d\left( [m, \alpha \chi_p(\sigma)] \cdot \chi_{1,p}(\xi_m^a \gamma)(\sigma) \right)(b)
\]
where the last equality follows as the measure \( [m, \alpha \chi_p(\sigma)] \cdot \chi_{1,p}(\xi_m^a \gamma) \) is supported on \( a \chi_p(\sigma) + m\mathbb{Z}_p \subset \mathbb{Z}_p \). In the same way, we get that
\[
(5.9) \quad m^{k-1}\text{Li}_k(\xi_m^a)^{\gamma x - a}(\sigma) = \frac{m^{k-1}}{(k-1)!} \int_{\mathbb{Z}_p} (v - \alpha \chi_p(\sigma))^{k-1} d\chi_{1,p}(0\overline{1}^{-\gamma} \xi_m^a)(\sigma)(v)
\]
\[
= -(-1)^{k-1} \frac{m^{k-1}}{(k-1)!} \int_{\mathbb{Z}_p} (v + \alpha \chi_p(\sigma))^{k-1} d\left( \chi_{1,p}(0\overline{1}^{-\gamma} \xi_m^a \gamma)(\sigma) \right)(v)
\]
\[
= \frac{(-1)^k}{(k-1)!} \int_{\mathbb{Z}_p} b^{k-1} d\left( [m, \alpha \chi_p(\sigma)] \cdot (\chi_{1,p}(\xi_m^a \gamma)(\sigma)) \right)(b).
\]

Now, we enter the situation of Theorem 1.1 and §3, i.e., \( a, m \in \mathbb{Z} \) (\( m > 1 \)) are integers with \( m \nmid a \), and set \( \gamma := \Gamma_{a/m} \), \( \alpha := a/m \).

Corollary 5.5. For the adelic Hurwitz measure \( \hat{\chi}_{a,m} = [m, \alpha \chi_p(\sigma)] \cdot \xi_{a/m} \in \hat{\mathbb{Z}}[[\hat{\mathbb{Z}}]] \), the \( p \)-adic image \( \hat{\chi}_{p,a,m}(\sigma) \in \mathbb{Z}_p[[\mathbb{Z}_p]] \) satisfies
\[
\int_{\mathbb{Z}_p} b^{k-1} d\hat{\chi}_{p,a,m}(\sigma)(b) = \frac{m^{k-1}}{k} B_k \left( \frac{a}{m} \right) (1 - \chi_p(\sigma)^k) \quad (\sigma \in G_{Q(\mu_m)}, \ k \geq 2).
\]

Proof. Combining the above calculations (5.8) and (5.9), we obtain from Theorem 5.1:
\[
\frac{m^{k-1}}{k} B_k(\alpha)(1 - \chi_p(\sigma)^k) = m^{k-1} \left( \text{Li}_k(\xi_m^{-a} \gamma x^a)(\sigma) + (-1)^k \text{Li}_k(\xi_m^{-a} \gamma x - a)(\sigma) \right)
\]
\[
= \int_{\mathbb{Z}_p} b^{k-1} \left( [m, \alpha \chi_p(\sigma)] \cdot (\chi_{1,p}(0\overline{1}^{-\gamma} \xi_m^{-a} \gamma)(\sigma)) (b) + \chi_{1,p}(0\overline{1}^{-\gamma} \xi_m^{-a} \gamma)(\sigma) \right)(b) \]
where \( \zeta_{p,a/m}(\sigma) \) is the image of \( \zeta_{a/m}(\sigma) \) by the projection \( \hat{Z}[[\hat{Z}]] \to \mathbb{Z}_p[[\mathbb{Z}_p]] \). This concludes the proof of the corollary.

5.3. **Proof of Theorem 1.1.** Note first that the support of the measure \( \hat{\zeta}_{p,a,m}(\sigma) \) is \( a\chi_p(\sigma) + m\mathbb{Z}_p \). When \( p \mid m \) and \( p \nmid a \), it is included in \( \mathbb{Z}_p^* \) so that the above Corollary proves the case.

**Remark 5.6.** It is worth noting that we do not need to assume \( 0 < a < m \) for the construction of the measure \( \hat{\zeta}_{a,m} \) and the integration property in the above case of \( p \mid m \). This leads to Remark 1.3 of Introduction.

The case \( p \nmid m \) was treated [W3] in the setting where pro-\( p \) path \( \gamma \) is taken suitably for a fixed \( p \). In our present case, we are taking \( \gamma \) to be the topological path \( \Gamma_{a/m} : \overline{01} \to \mathbb{C}_m \) (cf. Remark 5.4). We also need the assumption \( 0 < a < m \) for the following

**Lemma 5.7.** Given \( m, a \in \mathbb{Z}, m > 1 \) as in Theorem 1.1, suppose that a prime \( p \) does not divide \( m \). Let \( a_1, \delta \in \mathbb{Z} \) be integers such that \( a = pa_1 + \delta m \) with \( 1 \leq a, a_1 < m \). Then,

(i) \( (\Gamma_{a/m})_{(\delta)}^{(-\delta)} = \Gamma_{a_1/m} \).

(ii) \( (\Gamma_{a/m})_{(\delta)}^{(\delta)} = \Gamma_{a_1/m} \).

(iii) \( \zeta_{a,m}(\sigma) = [p, -\delta\chi(\sigma)] \zeta_{a_1,m}(\sigma) \) for \( \sigma \in \mathbb{G}_{(\mu m)} \).

**Proof.** (i) results from a good compatibility of our topological paths \( \Gamma_{a/m} \) introduced in §3.1 with the lifting along \( V_r \to V_1 \). Indeed,

\[
(\Gamma_{a/m} \cdot x^{-\delta})_p = (\Gamma_{a_1/m} \cdot x^{-\delta})_p = \Gamma_{a_1/m} \]

which derives (i). For (ii), suppose \( 1 \leq a, a_1 < m \). Then, noting that \( \tilde{\Gamma}_{a/m}, \tilde{\Gamma}_{a_1/m} \) are homotopic to the complex conjugates of \( \Gamma_{a/m}, \Gamma_{a_1/m} \) respectively (cf. Remark 3.2), we have

\[
(\tilde{\Gamma}_{a/m} \cdot x^{\delta})_p = (\tilde{\Gamma}_{a_1/m} \cdot x^{\delta})_p = \Gamma_{a_1/m} \]

This derives (ii). Finally, using (3.9), we see from (i) and (ii):

\[
\zeta_1(\Gamma_{a/m})(\sigma) = [p, \delta\chi(\sigma)] \zeta_1(\Gamma_{a_1/m}),
\]

\[
\zeta_1(\Gamma_{a/m})(\sigma) = [p, -\delta\chi(\sigma)] \zeta_1(\Gamma_{a_1/m}),
\]

hence from (3.10) we find

\[
[p, -\delta] \zeta_{a_1,m}(\sigma) = [p, -\delta] \left( [p, \delta\chi(\sigma)] \zeta_1(\Gamma_{a_1/m})(\sigma) + \epsilon \cdot \zeta_1(\Gamma_{a_1/m})(\sigma) \right)
\]

\[
= [p, -\delta] \zeta_1(\Gamma_{a_1/m})(\sigma) + \epsilon \cdot [p, \delta] \zeta_1(\Gamma_{a_1/m})(\sigma)
\]

\[
= \zeta_1(\Gamma_{a_1/m})(\sigma) + \epsilon \cdot \zeta_1(\Gamma_{a_1/m})(\sigma)
\]

\[
= \zeta_{a_1,m}(\sigma).
\]

This settles the proof of (iii). \( \square \)

Now, we compute the target integral of Theorem 1.1 in the case \( p \nmid m \):

\[
\int_{\mathbb{Z}_p} b^{k-1} d[m, a\chi_p(\sigma)] \zeta_{p,a/m}(\sigma)(b)
\]

\[
= \int_{\mathbb{Z}_p} b^{k-1} d[m, a\chi_p(\sigma)] \zeta_{p,a/m}(\sigma)(b) - \int_{\mathbb{Z}_p} b^{k-1} d[m, a\chi_p(\sigma)] \zeta_{p,a/m}(\sigma)(b),
\]

where the first term is calculated as:

\[
(5.10) \quad \int_{\mathbb{Z}_p} b^{k-1} d[m, a\chi_p(\sigma)] \zeta_{p,a/m}(\sigma)(b) = \frac{m^{k-1}}{k} B_k \left( \frac{a}{m} \right) (1 - \chi_p(\sigma)^k)
\]
by Corollary 5.5. For the second term, we observe:

$$\int_{\mathbb{Z}_p} b^{k-1}d[m, a\chi_p(\sigma)]_s \zeta_{p,a/m}(\sigma)(v) = \int_S (mv + a\chi_p(\sigma))^{k-1}d\zeta_{p,a/m}(\sigma)(v)$$

with $S := \{v \in \mathbb{Z}_p \mid mv + a\chi_p(\sigma) \in p\mathbb{Z}_p\}$. Since $p \nmid m$, we can choose integers $a_1, \delta \in \mathbb{Z}$ such that $a = a_1p + m\delta$. We set $a_1 = (ap^{-1})$ to be the least positive one as introduced in Theorem 1.1. In this set up, the condition $mv + a\chi_p(\sigma) = m(v + \delta\chi_p(\sigma)) + pa_1\chi_p(\sigma) \in p\mathbb{Z}_p$ is equivalent to the condition $v + \delta\chi_p(\sigma) + p\beta (\beta \in \mathbb{Z}_p)$, then $mv + a\chi_p(\sigma) = p(m\beta + a_1)$. Noting that Lemma 5.7 (iii) implies $\zeta_{p,a/m}(\sigma) = [p, -\delta\chi(\sigma)] \zeta_{p,a_1,m}(\sigma)$ for the $p$-adic images of measures by $\hat{\mathbb{Z}}[[\hat{\mathbb{Z}}]] \to \mathbb{Z}_p[[\mathbb{Z}_p]]$, we obtain

$$\int_S (mv + a\chi_p(\sigma))^{k-1}d\zeta_{p,a/m}(\sigma)(v) = p^{k-1} \int_{\mathbb{Z}_p} b^{k-1}d[m, a\chi_p(\sigma)]_s \zeta_{p,a_1,m}(\sigma)(b)$$

$$= p^{k-1} \int_{\mathbb{Z}_p} b^{k-1}d[m, a\chi_p(\sigma)]_s \zeta_{p,a_1,m}(\sigma)(b)$$

$$= p^{k-1} \int_{\mathbb{Z}_p} b^{k-1}d\hat{\zeta}_{p,a_1,m}(\sigma)(b)$$

$$= \frac{(pm)^{k-1}}{k} B_k \left( \frac{a_1}{m} \right) (1 - \chi_p(\sigma)^k),$$

where the last identity follows from Corollary 5.5. This, combined with (5.10), settles the remained case of Theorem 1.1.

□

Appendix A. Cohen’s $p$-adic Hurwitz zeta function

In this appendix, we shall relate the $p$-adic Hurwitz zeta function $\zeta_p(s, x)$ introduced in H. Cohen’s book [Co] to the $p$-adic Hurwitz zeta function of Shiratani type ([Sh]) discussed in our main text. Let $p$ be a prime, and let $q = p$ or $q = 4$ according as $p > 2, p = 2$ respectively. Set $C\mathbb{Z}_p := \mathbb{Q}_p \setminus \frac{p}{q}\mathbb{Z}_p$. Cohen’s $\zeta_p(s, x)$ is defined first in [Co, §11.2.2] for $x \in C\mathbb{Z}_p$ and is then defined also for $x \in \mathbb{Z}_p$ in [Co, §11.2.4]. Our main goal is to give connection with $\zeta_p(s, \frac{a}{m})$ in the formulas (A.4) and (A.9) below.

The case $\zeta_p(s, x)$ for $x \in C\mathbb{Z}_p$:

In [Wa, Theorem 5.9], a $p$-adic meromorphic function $H_p(s, a, m)$ in $s$ is introduced for a pair of integers $a, m$ with $q \mid m$, $p \nmid a$. It satisfies

$$(A.1) \quad H_p(1 - k, a, m) = -\omega(a)^{-k} \frac{m^{k-1}}{k} B_k \left( \frac{a}{m} \right) \quad (k \in \mathbb{N}),$$

where $\omega : \mathbb{Z}_p^\times \to \mu_p$ ($e := |(\mathbb{Z}/q\mathbb{Z})^\times|$) is the $p$-adic Teichmüller character. Cohen extends $\omega$ to $\omega_v : \mathbb{Q}_p^\times \to \mathbb{F}_p \cdot \mu_p$ by $\omega_v(u) = p^n\omega(u)$ ($u \in \mathbb{Z}_p^\times, n \in \mathbb{Z}$). Then, the interpolation property of $\zeta_p(s, x)$ for $x \in C\mathbb{Z}_p$ given in [Co, Theorem 11.2.9] reads

$$\zeta_p(1 - k, x) = -\omega_v(x)^{-k} \frac{B_k(x)}{k} \quad (k \in \mathbb{N})$$

which specializes for $x = a/m \in \mathbb{Q} \cap C\mathbb{Z}_p$ ($p \nmid a, q \mid m$) to

$$(A.2) \quad \zeta_p(1 - k, \frac{a}{m}) = \omega_v(m) \cdot H_p(1 - k, a, m)$$

$$= -\omega_v(m) \cdot \omega(a)^{-k} \frac{m^{k-1}}{k} B_k \left( \frac{a}{m} \right)$$
Restricting \( k \) to those positive integers in a same class in \( \mathbb{Z}/e\mathbb{Z} \), we obtain a relation between special values of \( \zeta_p(s,a/m) \) and \( \{L_p^{[\beta]}(s;a,m)\}_{\beta \in \mathbb{Z}/e\mathbb{Z}} \) of Remark 1.4 as follows:

\[(A.3) \quad \zeta_p(1-k, \frac{a}{m}) = -\left(\frac{\omega_r(m)}{\omega(a)^\beta}\right) L_p^{[\beta]}(1-k; a, m) \quad (k \equiv \beta(\text{mod } e), \ k \geq 1).\]

Hence, under the assumption \( q \mid m \) and \( p \nmid a \), for any \( \beta \in \mathbb{Z}/e\mathbb{Z} \), it follows that

\[(A.4) \quad \zeta_p(s, \frac{a}{m}) = -\left(\frac{\omega_r(m)}{\omega(a)^\beta}\right) L_p^{[\beta]}(s; a, m)\]

for \( s \) in the space \( \beta + \frac{a}{p}\mathbb{Z}_p \) which is one of the forms \( \mathbb{Z}_p, (p > 2), 2\mathbb{Z}_2 \) or \( 1 + 2\mathbb{Z}_2 \). Due to Remark 1.3, this formula holds true for all \( a \in \mathbb{Z} \) with \( m \nmid a \).

**The case \( \zeta_p(s, x) \) for \( x \in \mathbb{Z}_p \):**

In this case, let us first observe the following identity:

\[(A.5) \quad \zeta_p(1-k, x) = \frac{-1}{k} B_k(\tilde{\omega}^{-k}, x) \quad (k \in \mathbb{Z}_{\geq 1})\]

where \( \tilde{\omega} \) is the Teichmüller character on \( \mathbb{Z}_p \) (extended by 0 on \( q\mathbb{Z}_p \)) and \( B_k(\chi, *) \) is the \( \chi \)-Bernoulli polynomial defined in \([\text{Co}, \S 9.4.1]\).

**Proof.** Write \( x = p^n\alpha \in \mathbb{Z}_p \) with \( p \nmid \alpha \) and set \( N = p^n = p^{n+1} \). We make use of Corollary 11.2.15 of \([\text{Co}]\) and the notations there. As \( \omega_r(N) = p^n \), \( \langle N \rangle = 1 \), we have for \( s = 1-k \):

\[p^n \cdot \zeta_p(1-k, x) = \sum_{0 \leq j < p^n \atop p \nmid j} \zeta_p(1-k, \frac{x}{p^n} + \frac{j}{p^n}).\]

In RHS here, it follows from \([\text{Co}]\) Theorem 11.2.9 that

\[\zeta_p(1-k, \frac{x}{p^n} + \frac{j}{p^n}) = -(p^{-n}\omega(j))^{-k} \frac{1}{k} B_k(\frac{x}{p^n} + \frac{j}{p^n}).\]

Hence

\[p^n \cdot \zeta_p(1-k, x) = -\frac{p^{kv}}{k} \cdot \sum_{j=0}^{p^n} \tilde{\omega}^{-k}(j) \cdot B_k(\frac{x}{p^n} + \frac{j}{p^n})\]

\[= -\frac{p^{kv}}{k} \cdot p^{(1-k)} B_k(\tilde{\omega}^{-k}, x) \quad (\text{by } [\text{Co}] \text{ Lemma 9.4.7}).\]

This proves \((A.5)\).

Let \( e = |(\mathbb{Z}/q\mathbb{Z})^\times| \). Then,

\[(A.6) \quad \zeta_p(1-k, x) = -\frac{1}{k} \left(B_k(x) - p^{k-1} B_k(\frac{x}{p})\right)\]

for \( k \in \mathbb{Z}_{\geq 1} \) and \( k \equiv 0 \mod e \).

**Proof.** When \( k \equiv 0 \mod e \), we have \( \tilde{\omega}^k(j) = 0, 1 \) according as \( p \mid j, p \nmid j \). Putting this into the basic identity of \( \chi \)-Bernoulli polynomial \([\text{[Co] Prop. 9.4.5}]\), we find

\[B_k(\tilde{\omega}^{-k}, x) = p^{k-1} \sum_{j=0}^{p-1} \tilde{\omega}(j)^{-k} B_k(\frac{x+j}{p}) = p^{k-1} \sum_{j=1}^{p-1} B_k(\frac{x+j}{p}).\]
On the other hand, from the usual distribution formula of the Bernoulli polynomial: it follows that
\[ B_k(x) = p^{k-1} \sum_{j=0}^{p-1} B_k\left(\frac{x}{p} + \frac{j}{p}\right). \]

By comparing these two identities, we obtain (A.6).

Suppose \( x = \frac{a}{m} \in \mathbb{Q} \cap \mathbb{Z}_p \) with \( p \nmid m \). Observe that the above interpolation property of Cohen’s \( \zeta_p(s, x) \) (A.6) reads then
\[
\zeta_p(1-k; \frac{a}{m}) = -\frac{1}{k} \left( B_k\left(\frac{a}{m}\right) - p^{k-1} B_k\left(\frac{a}{mp}\right) \right) \quad (k \geq 1).
\]

It is not straightforward to find a connection from this to the interpolation property of Shiratani’s \( \zeta^{Sh}_p(s, a, m) = -L_p^{[0]}(s; a, m) \) (cf. Remark 1.4):
\[
\zeta_p^{Sh}(1-k; a, m) = -\frac{m^{k-1}}{k} \left( B_k\left(\frac{a}{m}\right) - p^{k-1} B_k\left(\frac{(ap^{-1})}{m}\right) \right)
\]
for all \( k \in \mathbb{Z}_{>0}, k \equiv 0 \mod e \), where \( a \) and \( (ap^{-1}) \) are supposed to be positive integer \(< m\) such that \( (ap^{-1})p \equiv a \mod m \). To connect (A.7) and (A.8), let \( r \) be the unique integer with \( a + mr = (ap^{-1})p \) so that \( \frac{a+mr}{mp} = \frac{(ap^{-1})}{m} \). Note that \( r > 0 \), due to the assumption \( 0 < a, (ap^{-1}) < m \). Then, replacing \( a \) by \( a + mr \) in (A.7), we find
\[
\zeta_p(1-k; \frac{a+mr}{m}) - m^{-k} \zeta_p^{Sh}(1-k; a, m) = \frac{1}{k} B_k(r + \frac{a}{m}) - \frac{1}{k} B_k\left(\frac{a}{m}\right)
= \sum_{v=0}^{r-1} \left(\frac{a+mv}{m}\right)^{k-1}
\]
for all \( k > 0, k \equiv 0 \mod e \). We claim below that \( p \nmid (a+mv) \) for all \( v \in [0, r-1] \) so that the existence of \( p \)-adic analytic functions \( \left(\frac{a}{m} + v\right)^s \) provides a connection between Cohen’s \( L_p(s, x) \) with \( x \in \mathbb{Z}_p \) and Shiratani’s \( \zeta^{Sh}_p(s; a, m) = -L_p^{[0]}(s; a, m) \), namely, it holds that
\[
\zeta_p(s, \frac{a + mr}{m}) = m^s \cdot \zeta^{Sh}_p(s; a, m) + \sum_{v=0}^{r-1} \left(\frac{a+mv}{m}\right)^{-s}
\]
for \( s \in \beta + \frac{r}{2} \mathbb{Z}_p \), under the assumptions \( p \nmid m, 0 < a < m, p \mid (a+mr) \) and \( 0 < a + mr < pm \). The assertion (A.9) is thus reduced to the following elementary

Claim. Notations being as above, let \( r_0 \) be the least nonnegative integer such that \( a + mr_0 \equiv 0 \mod p \). Then, \( a + mr_0 = (ap^{-1})p \).

Proof. If \( r_0 \geq p \), then \( a + m(r_0 - p) \equiv a + m(r_0) \equiv 0 \mod p \), which contradicts the minimality of \( r_0 \). Therefore \( r_0 < p \). If \( r_0 = p-1 \), then writing \( a + m(p-1) = xp \), we have \( p(m-x) = m-a > 0 \). Hence \( m > x > 0 \), i.e., \( x = (ap^{-1}) \). Assume that \( r_0 < p-1 \). If \( mp \leq a + mr_0 \), then since \( a < m \), it follows that \( 0 \leq a + m(r_0 - p) < m(r_0 - p + 1) \), hence that \( p - 1 \leq r_0 \) contradicting the assumption. Thus \( mp > a + mr_0 \). Writing \( a + mr_0 = xp \), we obtain \( m > x \), i.e., \( x = (ap^{-1}) \).

Example. Let \( p = 11, a = 3 \) and \( m = 106 \). Noting \( 106 \equiv 7 \mod 11 \) and \( 3 + 7 \cdot 9 = 61 \), one finds \( 3 + 106 \cdot 9 = 957 = 87 \cdot 11 \). Hence \( (3 \cdot 11^{-1}) = 87 \). Now, the core sum in the above construction reads \( \sum_n \left(\frac{n}{106} + v\right)^{k-1} = \left(\frac{3}{106}\right)^{k-1} + \left(\frac{109}{106}\right)^{k-1} + \left(\frac{215}{106}\right)^{k-1} + \left(\frac{321}{106}\right)^{k-1} + \left(\frac{427}{106}\right)^{k-1} + \left(\frac{109}{106}\right)^{k-1} + \left(\frac{215}{106}\right)^{k-1} + \left(\frac{321}{106}\right)^{k-1} + \left(\frac{427}{106}\right)^{k-1} \). There do exist \( 11 \)-adic analytic functions that interpolate \( \left(\frac{3}{106} + v\right)^{k-1} \) at \( s = 1-k (k \equiv 0 \mod 10) \) for \( v = 0,1,\ldots,8 \) respectively.
**Question.** It is unclear if $L_p^{[\beta]}(s; a, m)$ for $p \nmid m$, $\beta \neq 0$ ($e$) can be expressed in terms of Cohen’s $\zeta_p(s, x)$.

**Appendix B. Path conventions**

In this Appendix, we quickly summarize two conventions on étale paths mostly used in our papers. Just for simplicity, we call one system of conventions the traditional form (‘$t$-form’) and another system the electronic form (‘$e$-form’). The present paper and most papers by the second author obey the $t$-form, whereas most papers by the first author and our previous common papers [NW1-3] obey the $e$-form. The purpose of this Appendix is to serve a dictionary to translate formulas between these two forms.

Let $\mathcal{C}$ be a Galois category, for example, that of the finite étale covers of an algebraic variety. We write $a, b, c, \ldots$ for general symbols playing roles of base points for $\pi_1(\mathcal{C})$ and $\omega_a, \omega_b, \omega_c, \ldots$ for the corresponding Galois functors $\mathcal{C} \to \text{Sets}$. The path space between two points $a$ and $b$ is by definition the set $\text{Isom}(\omega_a, \omega_b)$ whose element is a compatible family of isomorphisms of fiber sets $\gamma_U : \omega_a(U) \xrightarrow{\sim} \omega_b(U)$ over $U \in \text{Ob}(\mathcal{C})$. In $t$-form, an element $\gamma$ of $\text{Isom}(\omega_a, \omega_b)$ is called a (t-)path from $a$ to $b$ and written as $\gamma : a \rightarrow b$. In $e$-form, the same $\gamma \in \text{Isom}(\omega_a, \omega_b)$ is called an $e$-path from $b$ to $a$ and written as $\gamma : b \rightarrow a$. Remind that, for each $U \in \text{Ob}(\mathcal{C})$, $\gamma_U(s)$ is defined for elements $s \in \omega_a(U)$. In $e$-form, we may imagine that the waving arrow $\gamma : b \rightarrow a$ flows like an electronic current that conveys electron $s \in \omega_a(U)$ back into $\omega_b(U)$.

We shall make use of the notation

$$\pi(\mathcal{C}; b, a) := \text{Isom}(\omega_a, \omega_b)$$

(B.1)

to designate the set of $t$-paths from $a$ to $b$ as well as the set of $e$-paths from $b$ to $a$. Accordingly, if $\gamma_1 \in \text{Isom}(\omega_a, \omega_b)$ and $\gamma_2 \in \text{Isom}(\omega_b, \omega_c)$, then the composite $\gamma_2 \gamma_1 \in \text{Isom}(\omega_a, \omega_c)$ is defined. We have

(B.2) $$\gamma_1 \gamma_2 \cdot [a \rightarrow b] = [b \rightarrow c] \cdot [a \rightarrow b] \quad \text{(viz. } [c \leftarrow b] \cdot [b \rightarrow a])$$

(B.3) $$[c \rightarrow b] \cdot [b \rightarrow a].$$

Next, let $F$ be a subfield of $\mathcal{C}$ and let $\mathcal{C}$ be the Galois category of finite étale covers of an algebraic variety $V$ over $F$. If $a$ is an $F$-rational (tangential) points on $V$, then the sequence of finite sets $\{\omega_a(U)\}_{U \in \text{Ob}(\mathcal{C})}$ have compatible actions by $GF$, which defines the map $GF \rightarrow \text{Isom}(\omega_a, \omega_a) = \pi_1(V, a)$. For two such points $a, b$, we define the canonical left $GF$-action on $\text{Isom}(\omega_a, \omega_b)$ by $\gamma \mapsto \sigma \gamma \gamma^{-1}$ ($\sigma \in GF$). Observe that, concerning Galois actions, no difference occurs between $t$-form and $e$-form.

Suppose now that $V = P^1_\mathbb{Q} \setminus \{0, 1, \infty\}$. Denote by $x, y$ the standard loops based at $01$ running around the punctures 0, 1 respectively with anticlockwise $t$-arrows $\rightarrow$, and let $x, y$ be those loops with anticlockwise $e$-arrows $\sim$. Then, $x = x^{-1}, y = y^{-1}$. Let $z$ be a $F$-rational (tangential) point on $V$. For a $t$-path $\gamma : 01 \rightarrow z$ on $V \otimes F$, we define a Galois associator in $t$-form by

(B.4) $$f_\gamma(\sigma) := \gamma^{-1} \cdot \sigma(\gamma) \in \pi_1^{et}(V, 01) \quad (\sigma \in GF)$$

as in §2 of the present paper, whereas, for an $e$-path $\delta : 01 \sim z$ on $V \otimes F$, we define another Galois associator in $e$-form by

(B.5) $$f^e_\delta(\sigma) := \delta \cdot \sigma(\delta)^{-1} \in \pi_1^{et}(V, 01) \quad (\sigma \in GF)$$

as in [NW1]. Therefore, assuming $\delta = \gamma^{-1}$, we find

(B.6) $$f_\gamma(\sigma) = f^e_\delta(\sigma) \quad (\sigma \in GF).$$
Given a prime $\ell$, let $\pi_{Q_\ell}$ be the pro-unipotent completion of the maximal pro-$\ell$ quotient of $\pi_1(V_{\overline{Q_\ell}}, \overline{0})$. Consider the above $x, y, x, y$ as $Q_\ell$-loops based at $\overline{0}$ on $V$, and regard $\gamma : \overline{0}1^{\to}z$ and $\delta : \overline{0}1^{\leftrightarrow}z$ as $Q_\ell$-paths on $V$. If $\delta = 1^{-1}$, then $(\gamma x^\alpha)^{-1} = x^\alpha \delta$; hence it follows from (B.6) that

$$f_{x^\alpha}(\sigma) = f_{x^\alpha}^\delta \quad (\delta = 1^{-1}, \sigma \in G_F, \alpha \in Q_\ell).$$

(B.7)

Now, let us compare $\ell$-adic Galois polylogarithms in $t$-form and $e$-form. Define generators $X, Y$ (resp. $X, Y$) of Lie($\pi_{Q_\ell}$) so that $e^X, e^Y$ (resp. $e^X, e^Y$) are the $\ell$-adic images of $x, y$ (resp. $x, y$) in $\pi_{Q_\ell}$. Then $X = -\overline{X}, Y = -\overline{Y}$. Let $I_Y$ (resp. $I_\ell$) denote the ideal of Lie($\pi_{Q_\ell}$) generated by those Lie words in $X, Y$ (resp. $X, Y$) containing $Y$ (resp. $Y$) twice or more. Obviously we have $I_Y = I_\ell \subset \text{Lie}(\pi_{Q_\ell})$. In $e$-form, we have the Lie expansion

$$\log(f_{x^\alpha}(\sigma)) \equiv \rho_{z, \gamma}(\sigma)X + \sum_{k=1}^{\infty} \ell_i(z; \gamma)(\sigma)(\text{ad} X)^{k-1}(\overline{Y}) \quad \text{mod } I_Y$$

extending [NW2, Definition 5.4] to any $Q_\ell$-paths $\delta : \overline{0}1^{\leftrightarrow}z$. Note that interpretation of $\rho_{z, \gamma}$ as a Kummer 1-cocycle along power roots of $z$ is basically available only when $\gamma$ is a pro-$\ell$ path. On the side of $t$-form, one also has

$$\log(f_{x^\alpha}(\sigma)) \equiv \rho_{z, \gamma}(\sigma)X + \sum_{k=1}^{\infty} \ell_i(z; \gamma)(\sigma)[..[Y, X, \ldots, X] \quad \text{mod } I_Y$$

extending [W1, Definition 11.0.1] for any $Q_\ell$-path $\gamma : \overline{0}1^{\to}z$. Comparing (B.8) and (B.9) under the situation (B.6), we see that the $\rho_z$ and the $\ell$-adic polylogarithms $\ell_m(z)$ (written also as $\ell_m(z)$ in older papers) for $\gamma : \overline{0}1^{\to}z$ in $t$-form and for $\delta : \overline{0}1^{\leftrightarrow}z$ in $e$-form coincide with each other as functions on $G_F$ as long as $\delta = 1^{-1}$, i.e.,

$$\rho_{z, \gamma}(\sigma) = \rho_{z, \delta}(\sigma), \ell_i(z; \gamma)(\sigma) = \ell_i(z; \delta)(\sigma) \quad (\sigma \in G_F, k \geq 1, \delta = 1^{-1}).$$

(B.10)

Next, embed Lie($\pi_{Q_\ell}$) into the ring of non-commutative power series $Q_\ell[\![X, Y]\!] = Q_\ell[\![\overline{X}, \overline{Y}]\!]$ and expand $f_{x^\alpha}(\sigma) = f_{x^\alpha}$ into series in $X, Y$ or in $\overline{X}, \overline{Y}$. The coefficient at $YX^{k-1}$ appearing in the former expansion is the $\ell$-adic polylogarithm $\ell_i(z, \gamma)$ in §3 of this paper in $t$-form, while the coefficient at $\overline{Y}X^{k-1}$ in the latter expansion, which we denote by $\mathcal{L}i_k(\sigma, \delta)$ in $e$-form, was discussed in [NW3, Definition 6.2]. By definition, we have

$$\ell_k(\overline{0}1^{\to}z)(\sigma) = (-1)^k \mathcal{L}i_k(\overline{0}1^{\leftrightarrow}z)(\sigma) \quad (\sigma \in G_F, k \geq 1, \delta = 1^{-1}).$$

(B.11)

Finally, we recall from [NW3, Definition 6.4] the function

$$\tilde{X}^z_{\delta} : G_F \to Q_\ell$$

associated to any $Q_\ell$-path $\delta : \overline{0}1^{\leftrightarrow}z$ for $k \geq 1$ by the equation:

$$\tilde{X}^z_{\delta}(\sigma) = (-1)^{k+1}(k-1)! \sum_{i=1}^{k} \frac{\rho_{z, \delta}(\sigma)^{k-i}}{(k+1-i)!} \ell_i(z, \delta)(\sigma).$$

(B.12)

It is related to the above $\mathcal{L}i_k(\overline{0}1^{\leftrightarrow}z)(\sigma)$ by

$$\frac{-\tilde{X}^z_{\delta}}{(k-1)!} = \mathcal{L}i_k(\overline{0}1^{\leftrightarrow}z)(\sigma) \quad (\sigma \in G_F, k \geq 1).$$

(B.13)

When $\delta : \overline{0}1^{\leftrightarrow}z$ is a pro-$\ell$ path, then $\tilde{X}^z_{\delta}$ is the polylogarithmic character studied in [NW1] and is known to be valued in $\mathbb{Z}_\ell$ with explicit Kummer properties along a sequence of numbers.
For a path $\delta: \overrightarrow{01} \longrightarrow z$ in $t$-form, we employ the notation
\[ \tilde{X}_k(z, \gamma)(\sigma) := \tilde{X}_k^{\gamma, \delta}(\sigma) \quad (\sigma \in G_F) \]
where $\delta = \gamma^{-1}: \overrightarrow{01} \longrightarrow z$ is the corresponding path in $e$-form. It follows then that
\[ \frac{\tilde{X}_k(z, \gamma)(\sigma)}{(k-1)!} = (-1)^{k-1} \text{Li}_k(0\overrightarrow{11\gamma}z)(\sigma) \quad (\sigma \in G_F) \]
for any $\mathbb{Q}_\ell$-path $\gamma: \overrightarrow{01} \longrightarrow z$.

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