Classification of variations of mixed Hodge structures I

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The plan of the paper.

§0. Introduction.

0.1. The purpose of this note is to study a classification of variation of mixed Hodge structures over a given complex variety X. The starting point is the following well known result.

Let $\tilde{X} \to X$ be a universal covering space over X. The fundamental group $\pi_1(X, x)$ acts on $\tilde{X}$. Let V be a finite dimensional vector space over k and let $\rho : \pi_1(X, x) \to \mathrm{Aut}(X)$ be a representation. Then $\tilde{X} \times_\rho V \to X$ is a local system of k-vector spaces over X.

Theorem 0.1.1. *The category of local systems of finite dimensional k-vector spaces over X and the category of representations of $\pi_1(X, x)$ in finite dimensional k-vector spaces are equivalent.*

Let G be a complex affine algebraic group. Let $\pi : P \to X$ be a principal G-bundle equipped with an integrable connection ω. We assume that the image of the monodromy homomorphism $\Theta : \pi_1(X, x) \to G$ is Zariski dense in G. Let $\mathcal{O}[G]$ be the ring of regular functions on G. The group G acts on $\mathcal{O}[G]$ on the left by $({}^g f)(t) := f(g^{-1}t)$. The associated vector bundle is the bundle $\pi_* \mathcal{O}_P \to X$. We equip it with a connection ∇_ω induced by the connection ω. Let us denote the vector bundle $\pi_* \mathcal{O}_P \to X$ by $\mathcal{P}$. We assume that $\mathcal{P}$ carries a variation of Hodge structures of weight 0 (the connection is ∇_ω) and that $\mathcal{O}[G]$ caries a Hodge structure of weight 0. We assume that these Hodge structures are compatible with the multiplicative structures on $\mathcal{P}$ and on $\mathcal{O}[G]$, with the Hopf algebra structure on $\mathcal{O}[G]$ and with the (co-) action of $\mathcal{O}[G]$ on $\mathcal{P}$.

Definition 0.1.2. Let V be a finite direct sum of Hodge structures V_i of weight n_i. We say that

$$\rho : V \to \mathcal{O}[G] \otimes V$$

is a Hodge representation if ρ is a morphism of Hodge structures and

$$(id_{\mathcal{O}}[G] \otimes \rho) \circ \rho = (\mu \otimes id_V) \circ \rho,$$

where μ is induced by a multiplication $G \times G \to G$.

The group G acts on V ($G \ni g \to \rho(g) \in \mathrm{End}(V)$). We form an associated vector bundle $\mathcal{V}_\rho := (P \times_G V \to X)$. We shall see that $\mathcal{V}_\rho$ carries a variation of Hodge structures.

Consider the diagonal action of G on $\mathcal{P} \otimes V^*$. Then $(\mathcal{P} \otimes V^*)^G$ is the bundle $\mathcal{V}_\rho^*$. The bundles $\mathcal{P} \otimes V^*$ and $\mathcal{P} \otimes V^* \otimes \mathcal{O}[G]$ carry variations of Hodge structures. The action of G on $\mathcal{P}$ and V^* are given by Hodge representations (morphisms of variations of Hodge structures) $\mathcal{P} \to \mathcal{O}[G] \otimes \mathcal{P}$ and $V^* \to \mathcal{O}[G] \otimes V^*$. Hence the diagonal action of G on $\mathcal{P} \otimes V^*$ is given by a Hodge representation $\tau : \mathcal{P} \otimes V^* \to \mathcal{O}[G] \otimes \mathcal{P} \otimes V^*$. Observe that $(\mathcal{P} \otimes V^*)^G = \{x \in \mathcal{P} \otimes V^* \mid \tau(x) = 1 \otimes x\}$. This implies that the vector bundle $\mathcal{V}_\rho^* = (\mathcal{P} \otimes V^*)^G$ carries a variation of Hodge structures. Hence also $\mathcal{V}_\rho$ carries a variation of Hodge structures.

We denote by $HRep(\mathcal{O}[G])$ the category of Hodge representations of $\mathcal{O}[G]$. We denote by $V^{\mathcal{P}}_{HRep(\mathcal{O}[G])}$ the category of variations of Hodge structures of the form $\mathcal{V}_\rho$ ($\rho \in HRep(\mathcal{O}[G])$).

Theorem 0.1.3. *Assume that the Hodge structures $\mathcal{O}[G]$, V and the variation of Hodge structures $\mathcal{P}$ carry polarizations and the structure morphisms are compatible with polarizations . Then the categories $HRep(\mathcal{O}[G])$ and $V^{\mathcal{P}}_{HRep(\mathcal{O}[G])}$ are equivalent.*

0.2. We denote by $VMHS_{\mathcal{P}}$ the category of variations of mixed Hodge structures $\mathcal{V}$ over X such that $Gr^i_W \mathcal{V} \in V^{\mathcal{P}}_{HRep(\mathcal{O}[G])}$ for each i. Our aim is to classify such variations. First we construct a universal variation of mixed Hodge structures.

The inclusion of simplicial sets $\partial\Delta[1] \hookrightarrow \Delta[1]$ induces a morphism of cosimplicial spaces $p^\bullet : X^{\Delta[1]} \to X^{\partial\Delta[1]}$. Let $y \in X$. Let us set $(X, y)^{(\Delta[1],1)} := p^{\bullet -1}((y \times X)^\bullet)$, where $(y \times X)^\bullet$ is a constant cosimplicial space equal $y \times X$ in each degree. The sheaf $\mathcal{P}$ is a sheaf of $\mathcal{O}_X$-algebras and the connection $\nabla : \mathcal{P} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{P}$ is multiplicative. Hence the relative twisted De Rham complexes form a complex of sheaves $\Omega^* := \Omega_{(X,y)^{(\Delta[1],1)}/(y \times X)^\bullet}(\mathcal{P}^{\otimes(\bullet+1)})$ on $(X, y)^{(\Delta[1],1)}$. Applying a functor $Rp^\bullet_*$ we get a bicomplex of $\mathcal{O}_X$-modules on $y \times X$. Let $\mathfrak{t}$ be a functor which to a bicomplex associates its total complex.

Theorem 0.2.1. *We have*

i) The cohomology sheaves $H^i(\mathfrak{t}Rp^\bullet_*\Omega^*)$ carry a variation of mixed Hodge structures.

ii) $H^0(\mathfrak{t}Rp^\bullet_*\Omega^*)$ is a sheaf of $\mathcal{O}_X$-algebras and the mixed Hodge structure is compatible with multiplication;

iii) Let $\mathcal{H}$ be a fiber of $H^0(\mathfrak{t}Rp_*^\bullet\Omega^*)$ over (y,y). Then $\mathcal{H}$ is a Hopf algebra, $\mathcal{H}$ (co-)acts on $H^0(\mathfrak{t}Rp_*^\bullet\Omega^*)$ and the structure morphisms are morphisms of mixed Hodge structures.

0.3. Let $\mathcal{V} \in VMHS_{\mathcal{P}}$ and let V be a fiber of $\mathcal{V}$ over $y \in X$. Our aim is to construct a mixed Hodge representation

$$\tau_{\mathcal{V}} : V \to \mathcal{H} \otimes V.$$

The following observation is crucial in the construction of $\tau_{\mathcal{V}}$. Let $L(z)$ and $C(z)$ be in $\Omega^1(X) \otimes \mathrm{End}(V)$. Assume that $\varphi(z)$ satisfies an equation

$$d\varphi(z) + L(z)\varphi(z) = 0,\ \varphi(y) = Id_V.$$

Then the sum of iterated integrals

$$F(z) := \varphi(z)(Id + \int \varphi(z)^{-1}\circ(-C(z))\circ\varphi(z) + \int \varphi(z)^{-1}\circ(-C(z))\circ\varphi(z), \varphi(z)^{-1}\circ(-C(z))\circ\varphi(z) + \ldots)$$

satisfies an equation

$$dF(z) + (L(z) + C(z))F(z) = 0.$$

We assume for simplicity that the bundle $\mathcal{V}$ and the sub-bundles $W_i\mathcal{V}$ are trivial. The bundle $Gr_W\mathcal{V}$ is obtained from a Hodge representation $\tau_\varphi \in \mathcal{O}[G] \otimes \mathrm{End}(V)$. It is equipped with a connection form $L(z) \in \Omega^1(X) \otimes \mathrm{End}(V)$. The bundle $\mathcal{V}$ is equipped with a connection form $L(z) + C(z)$ and $C(z)$ vanishes when passing to the associated graded bundle. Let us set

0.3.1. $$\tau_{\mathcal{V}} := \tau_\varphi(g_0)^{-1} + \tau_\varphi(g_1)^{-1} \circ (-C(z_1)) \circ \tau_\varphi(g_1) \circ \tau_\varphi(g_0)^{-1} +$$

$$\tau_\varphi(g_2)^{-1} \circ (-C(z_2)) \circ \tau_\varphi(g_2) \circ \tau_\varphi(g_1)^{-1} \circ (-C(z_1)) \circ \tau_\varphi(g_1) \circ \tau_\varphi(g_0)^{-1} + \ldots$$

$$\in \oplus_{n=0}^\infty \Omega^n(y \times X^n \times y) \otimes \mathcal{O}[G]^{\otimes(n+1)} \otimes \mathrm{End}(V).$$

Theorem 0.3.2. *We have*

i) $\tau_{\mathcal{V}}$ is a cocycle (the corresponding cohomology class in $\mathcal{H} \otimes \mathrm{End}(V)$ we denote also by $\tau_{\mathcal{V}}$).

ii) $\tau_{\mathcal{V}} : V \to \mathcal{H} \otimes V$ is a mixed Hodge representation.

iii) If $\mathcal{V}$ and $\mathcal{V}'$ are isomorphic variations of mixed Hodge structures then $\tau_{\mathcal{V}} = \tau'_{\mathcal{V}} \in \mathcal{H} \otimes \mathrm{End}(V)$.

We denote by $MHRep(\mathcal{H})$ the category of mixed Hodge representations of $\mathcal{H}$. We define a functor $F : VMHS_{\mathcal{P}} \to MHRep(\mathcal{H})$ by $F(\mathcal{V}) := \tau_{\mathcal{V}}$. The functor F on morphisms is a restriction to fibers over y.

Theorem 0.3.3. *Let X be a smooth complex projective variety. Then the functor*

$$F : VMHS_{\mathcal{P}} \to MHRep(\mathcal{H})$$

is an equivalence of categories.

Let X be a complement of a divisor with normal crossings in a smooth complex projective variety. Let $DE(X, \mathcal{P})_{alg}$ be the category of algebraic vector bundles $\mathcal{V}$ over X equipped with a filtration $\{W_i\mathcal{V}\}$ and a regular integrable connection compatible with the filtration such that the associated graded vector bundle equipped with the induced connection is isomorphic to $P \times_{\varphi} V$ for some representation $\varphi : G \to \mathrm{Aut}(V)$.

Theorem 0.3.4. *The functor $F : DE(X, \mathcal{P})_{alg} \to Rep(\mathcal{H})$ is an equivalence of categories.*

§1. Hodge representations.

1.0. Let G be a reductive algebraic group over a field of complex numbers C and let $\mathcal{O}[G]$ be a Hopf algebra of regular functions on G with structure morphisms $\mu : \mathcal{O}[G] \to \mathcal{O}[G] \otimes \mathcal{O}[G]$, $\iota : \mathcal{O}[G] \to \mathcal{O}[G]$ and $\varepsilon : \mathcal{O}[G] \to C$. We assume that $\mathcal{O}[G]$ carries a Hodge structure of weight 0 and that the structure morphisms and the multiplication $m : \mathcal{O}[G] \otimes \mathcal{O}[G] \to \mathcal{O}[G]$ are morphisms of Hodge structures.

Let V be vector space over C. We say that $\mathcal{O}[G]$ acts on V if there is a linear map $\tau_V : V \to \mathcal{O}[G] \otimes V$ such that

$$(\mu \otimes id_V) \circ \tau_V = (id_{\mathcal{O}}[G] \otimes \tau_V) \circ \tau_V \ \text{ and } \ (\varepsilon \otimes id_V) \circ \tau_V = id_V.$$

We denote also by $\tau_V : G \to \mathrm{Aut}V$ an action of G on V induced by an action τ_V of $\mathcal{O}[G]$ on V. Let $g \in G$. Then the automorphism $\tau_V(g)$ is the composition $e_{g^{-1}} \circ \tau_V$, where $e_{g^{-1}} : \mathcal{O}[G] \to C$ is the evaluation map at g^{-1}.

Assume that V carries a Hodge structures of weight n. We say that $\tau : V \to \mathcal{O}[G] \otimes V$ is an irreducible Hodge representation if τ is a morphism of Hodge structures and if the representation $\tau : G \to \mathrm{Aut}V$ is irreducible. A direct sum of irreducible Hodge representations is called a Hodge representation of $\mathcal{O}[G]$. We denote by $HRep(\mathcal{O}[G])$ the category of Hodge representations of $\mathcal{O}[G]$.

Observe that any Hodge representation $\tau : V \to \mathcal{O}[G] \otimes V$ determines an element of Hodge type $(0,0)$ in $\mathcal{O}[G] \otimes V^* \otimes V = \mathcal{O}[G] \otimes \mathrm{End}(V)$. This element we shall also denote by τ.

1.1. Let X be a Zariski open in a smooth complex compact analytic variety. Let $\pi : P \to X$ be a principal G-bundle equipped with the integrable connection. Let $\Theta : \pi_1(X, x) \to G$ be the monodromy representation at $x \in X$. Assume that the image of Θ is Zariski dense in G. Assume that $\mathcal{P} := \pi_* \mathcal{O}_P \to X$ equipped with the connection deduced from the connection on $\pi : P \to X$ carries a variation of Hodge structures of weight 0 compatible with the action of $\mathcal{O}[G]$ equipped with a Hodge structure by 1.0. The choice of a point $x' \in \pi^{-1}(x)$ identifies $\mathcal{P}_x$, the fiber of $\mathcal{P}$ over x, with $\mathcal{O}[G]$.

We identify a vector bundle with its sheaf of sections.

Lemma 1.1.1. *Let G acts on $\mathcal{O}[G]$ on the left by $({}^g f)(t) := f(g^{-1}t)$. Then the associated vector bundle $P \times_G \mathcal{O}[G] \to X$ is $\pi_* \mathcal{O}_P \to X$.*

Proof. One cheks that the transition functions are the same for both bundles.

Let $\tau : V \to \mathcal{O}[G] \otimes V$ be a Hodge representation. Hence the group G acts on V by $\tau : G \to \mathrm{Aut} V$ and we can form an associated vector bundle $\mathcal{V}_\tau := (P \times_G V \to X)$. We shall see that $\mathcal{V}_\tau$ carries a variation of Hodge structures.

The group G acts on P on the right hence it acts on $\mathcal{P}$ on the left. G acts also on V^* on the left by $(gf)(v) := f(g^{-1}v)$. Hence G acts diagonally on $\mathcal{P} \otimes V^*$.

Lemma 1.1.2. *The vector bundle $\mathcal{V}_\tau$ is equal to the dual of the vector bundle $(\mathcal{P} \otimes V^*)^G$.*

Proof. We can suppose that the Hodge representation τ is irreducible. Let $G \times G$ acts on $\mathcal{O}[G]$ by $((h,g)f)(t) := f(h^{-1}tg)$ and on $V^* \otimes V$ by $(h,g)(f(-) \otimes v) := f(h^{-1}-) \otimes g(v)$. Then $V^* \otimes V$ is included in $\mathcal{O}[G]$ by a $G \times G$-equivariant map. Observe that for the diagonal action of G on $V^* \otimes V \otimes V^*$ given by $g(f_1 \otimes v \otimes f_2) := f_1 \otimes g(v) \otimes f_2 \circ g^{-1}$ we have $((V^* \otimes V) \otimes V^*)^G = V^* \otimes (V \otimes V^*)^G = V^* \otimes C = V^*$. Hence we get $(\mathcal{O}[G] \otimes V^*)^G = V^*$ and V^* is equipped with the action of G deduced from the action of G on $\mathcal{O}[G]$ given by $h(f)(t) = f(h^{-1}t)$. Therefore the transition functions of the bundles $\mathcal{V}_\tau^*$ and $(\mathcal{P} \otimes V^*)^G$ are the same.

Lemma 1.1.3. *If τ_V and τ_W are Hodge representations then $\tau_{V \otimes W}$ and τ_{V^*} are Hodge representations.*

Proof. Observe that $\tau_V \otimes \tau_W : V \otimes W \to (\mathcal{O}[G] \otimes V) \otimes (\mathcal{O}[G] \otimes W)$, $s : (\mathcal{O}[G] \otimes V) \otimes (\mathcal{O}[G] \otimes W) \to \mathcal{O}[G] \otimes \mathcal{O}[G] \otimes V \otimes W$ given by $s((f \otimes v) \otimes (g \otimes w)) = f \otimes g \otimes v \otimes w$ and $m \otimes id_{V \otimes W}$ are morphisms of Hodge structures. Hence their composition $\tau_{V \otimes W} := (m \otimes id_{V \otimes W}) \circ s \circ (\tau_V \otimes \tau_W)$ is also a morphism of mixed Hodge structures.

We shall show that τ_{V^*} is a Hodge representation. Let $\tau_V : V \to \mathcal{O}[G] \otimes V$. Then there is a finite dimensional Hodge structure $\mathcal{O}' \subset \mathcal{O}[G]$ such that τ_V factors through $V \to \mathcal{O}' \otimes V$. Passing to dual vector spaces we get morphisms of Hodge structures $(\mathcal{O}')^* \otimes V^* \to V^*$. Hence $\tau_{V^*} : V^* \to (\mathcal{O}')^{**} \otimes V^* = \mathcal{O}' \otimes V^* \subset \mathcal{O}[G] \otimes V^*$ is also a morphism of Hodge structures.

Lemma 1.1.4. *The vector bundle $\mathcal{V}_\tau$ carries a variation of Hodge structures.*

Proof. The actions of G on $\mathcal{P}$ and on V^* are given by a morphism of variations of Hodge structures $\mathcal{P} \to \mathcal{O}[G] \otimes \mathcal{P}$ and a Hodge representation $\tau_{V^*} : V^* \to \mathcal{O}[G] \otimes V^*$. Hence a diagonal action of G on $\mathcal{P} \otimes V^*$ is given by a morphism of variations of Hodge structures $\rho : \mathcal{P} \otimes V^* \to \mathcal{O}[G] \otimes \mathcal{P} \otimes V^*$. Let $\iota : \mathcal{P} \otimes V^* \to \mathcal{O}[G] \otimes \mathcal{P} \otimes V^*$ be given by $\iota(x) = 1 \otimes x$, where $x \in \mathcal{P} \otimes V^*$. Then ι is a morphism of variations of Hodge structures. Observe that $(\mathcal{P} \otimes V^*)^G = ker(\rho - \iota)$. This implies that the bundle $(\mathcal{P} \otimes V^*)^G$ and hence also the bundle $\mathcal{V}_\tau$ carry a variation of Hodge structures.

1.1.5. Let $x \in X$. Then the choice of a point $x' \in \pi^{-1}(x)$ defines a bijection of G onto $P_x := \pi^{-1}(x)$. Let us denote by $\mathcal{P}_x$ the fiber of $\mathcal{P}$ over x. Then $\mathcal{P}_x = \mathcal{O}[P_x]$. We assume that there is $x \in X$ and $x' \in \pi^{-1}(x)$ such that the induced morphism $\mathcal{P}_x \to \mathcal{O}[G]$ is an isomorphism of Hodge structures.

Let $\tau : V \to \mathcal{O}[G] \otimes V$ be a Hodge representation. Then the fiber of the variation of Hodge structures $\mathcal{V}_\tau$ over x is V.

1.2. Polarization (see [Sch] p.217 and 220).

We assume that the Hodge structures $\mathcal{O}[G]$ and V and the variation of Hodge structures $\mathcal{P}$ carry polarizations and the structure morphisms, multiplication, actions of $\mathcal{O}[G]$ are compatible with polarizations. Then $(\mathcal{P} \otimes V^*)^G$ also carries a polarization because it is a kernel of a morphism between polarized variations of Hodge structures. Hence $\mathcal{V}_\tau$ is a polarized variation of Hodge structures.

Further in this section a Hodge structure and a variation of Hodge structures are always polarized.

1.3. Let $\Theta : \pi_1(X, x) \to G$ be a homomorhism from 1.1. We denote by VHS_Θ the category of variations of Hodge structures $\mathcal{V}$ over X such that the monodromy homomorphism at x

$$\pi_1(X, x) \to \mathrm{Aut} V_x$$

is a composition

$$\tau_{\mathcal{V}} \circ \Theta : \pi_1(X, x) \to G \to \mathrm{Aut} V_x,$$

where $\tau_{\mathcal{V}} : V_x \to \mathcal{O}[G] \otimes V_x$ is a Hodge representation.

Theorem 1.3.1. *The categories $HRep(\mathcal{O}[G])$ and VHS_Θ are equivalent.*

Proof. The functor $\mathcal{V} : HRep(\mathcal{O}[G]) \to VHS_\Theta$ associates to a Hodge representation τ a variation of Hodge structures $\mathcal{V}_\tau$. Let $\mathcal{V} \in VHS_\Theta$ and let $\tau : V_x \to \mathcal{O}[G] \otimes V_x$ be a Hodge representation, which appears in the factorization of the monodromy homomorphism of $\mathcal{V}$ at the point x. The variation of Hodge structures $\mathcal{V}_\tau$ has the monodromy homomorphism at x given by $\tau \circ \Theta : \pi_1(X, x) \to G \to \mathrm{Aut}(V_x)$. Hence the monodromy homomorphisms of $\mathcal{V}$ and $\mathcal{V}_\tau$ are equal. It follows from [Sch] Theorem 7.24 that the variations of Hodge structures $\mathcal{V}$ and $\mathcal{V}_\tau$ are isomorphic.

It rests to show that the functor $\mathcal{V}$ on morphisms

$$\mathrm{Hom}_{HRep(\mathcal{O}[G])}(\tau, \tau') \to \mathrm{Hom}_{VHS_\Theta}(\mathcal{V}_\tau, \mathcal{V}_{\tau'})$$

is bijective. It is clear that it is injective. Let $f : \mathcal{V}_\tau \to \mathcal{V}_{\tau'}$ be a morphism of variation of Hodge structures. Restricting f to the fiber over x we get $f_x \in Rep_{\pi_1(X,x)}(V, V')$. The monodromy representations of $\pi_1(X, x)$ on V and V' factor through $\Theta : \pi_1(X, x) \to G$ because $\mathcal{V}_\tau$ and $\mathcal{V}_{\tau'}$ are associated vector bundles equipped with connections induced from the principal G-bundle $P \to X$. The image of Θ is Zariski dense in G. This implies that the diagram

$$\begin{array}{ccc} V & \xrightarrow{\tau} & \mathcal{O}[G] \otimes V \\ \Big\downarrow f_x & & \Big\downarrow id_{\mathcal{O}[G]} \otimes f_x \\ V' & \xrightarrow{\tau'} & \mathcal{O}[G] \otimes V' \end{array}$$

commutes. Let $\mathcal{V}(f_x) : \mathcal{V}_\tau \to \mathcal{V}_{\tau'}$ be a map induced by f_x. $\mathcal{V}(f_x)$ is a flat section of the variation of Hodge structures $\mathrm{Hom}(\mathcal{V}_\tau, \mathcal{V}_{\tau'})$ and it is of Hodge type (0,0) at x. It follows from [Sch] Corollary 7.23 that $\mathcal{V}(f_x)$ is a morphism of variations of Hodge structures. $\mathcal{V}(f_x)$ and f coincide at x, hence $\mathcal{V}(f_x) = f$ because two flat sections equal at one point are equal everywhere.

§2. Chain complexes.

2.0. Let X be a topological space. The inclusion of simplicial sets $\partial\Delta[1] \to \Delta[1]$ induces a morphism of cosimplicial spaces $p^\bullet : X^{\Delta[1]} \to X^{\partial\Delta[1]}$. Let $(x_2, x_1) \in X \times X$. We consider (x_2, x_1) as a constant cosimplicial space equal (x_2, x_1) in each degree. Then $p^{\bullet -1}(x_1, x_2)$ is a cosimplicial space equal $x_2 \times X^n \times x_1$ in degree n.

If Y is a smooth complex variety, we denote by $\Omega^*(Y)$ the De Rham complex of smooth complex valued differential forms on the smooth complex variety Y.

Let X be a smooth complex variety. Then $\oplus_{n=0}^{\infty}\Omega^*(x_2 \times X^n \times x_1)$ is a bicomplex with two commuting differentials. Let $T_1 := Tot(\oplus_{n=0}^{\infty}\Omega^*(x_2 \times X^n \times x_1))$ be the total complex of the bicomplex $\oplus_{n=0}^{\infty}\Omega^*(x_2 \times X^n \times x_1)$ (see [W1]).

2.1. Let $\partial : \Omega^*(X)^{\otimes n} \to \Omega(X)^{\otimes n}$ be the exterior differentiation. Let us define

$$\delta_i : \Omega^*(X)^{\otimes n} \to \Omega^*(X)^{\otimes (n-1)}$$

in the following way :

$$\delta_i(\omega_n \otimes \ldots \otimes \omega_1) := \omega_n \otimes \ldots \otimes \omega_{i+1} \wedge \omega_i \otimes \ldots \otimes \omega_1 \text{ for } i \neq 0, 1;$$

$$\delta_0(\omega_n \otimes \ldots \otimes \omega_1) := \omega_n \otimes \ldots \otimes \omega_2 \varepsilon_1(\omega_1);$$

$$\delta_n(\omega_n \otimes \ldots \otimes \omega_1) := \varepsilon_2(\omega_n)\omega_{n-1} \otimes \ldots \otimes \omega_1$$

where $\varepsilon_i(\omega) = e(x_i)$ if $\omega \in \Omega^0(X)$ and $\varepsilon_i(\omega) = 0$ if $\deg \omega > 0$. Let us set

$$\delta := \sum_{i=0}^{n} (-1)^i \delta_i : \Omega^*(X)^{\otimes n} \to \Omega(X)^{\otimes (n-1)}.$$

Then we have $\partial^2 = \delta^2 = \partial\delta - \delta\partial = 0$. Hence $\oplus_{n=0}^{\infty}\Omega^*(X)^{\otimes n}$ is a double complex. We define a total complex T_2 such that in degree k

$$(T_2)_k = Tot(\oplus_{n=0}^{\infty}\Omega^*(X)^{\otimes n})_k := \oplus_{k_1+\ldots+k_i-i=k}\Omega^{k_1}(X) \otimes \ldots \otimes \Omega^{k_i}(X).$$

If $\omega \in \Omega^{k_1}(X) \otimes \ldots \otimes \Omega^{k_i}(X)$ then $d(\omega) := \partial\omega + (-1)^{k_1+\ldots+k_i}\delta\omega$, where $k_i = deg\, \omega_i$.

The obvious map $T_2 \to T_1$ is a quasi-isomorphism.

2.2. We shall compare the complex $(Tot(\oplus_{n=0}^{\infty}\Omega^*(X)^{\otimes n}, d)$ with the bar construction on $\Omega^*(X)$, because certain constructions are more familier in terms of the bar construction. We set

$$B^{-n} = B^{-n}_{x_2,x_1} = B^{-n}_{x_2,x_1}(\Omega^*(X)) := \Omega^*(X)^{\otimes n}.$$

The differential $\partial' : B^{-n} \to B^{-n}$ is defined by

$$\partial'(\omega_1 \otimes \ldots \otimes \omega_n) := \sum_{i=1}^{n} (-1)^{k_1+\ldots+k_{i-1}+i} \omega_1 \otimes \ldots \otimes \partial\omega_i \otimes \ldots \otimes \omega_n.$$

The differential $\delta' : B^{-n} \to B^{-n+1}$ is defined by

$$\delta'(\omega_1 \otimes \ldots \otimes \omega_n) := \varepsilon_2(\omega_1)\omega_2 \otimes \ldots \otimes \omega_n + \sum_{i=1}^{n-1} (-1)^{k_1+\ldots+k_i+i} \omega_1 \otimes \ldots \otimes \omega_i \wedge \omega_{i+1} \otimes \ldots \otimes \omega_n +$$

$$(-1)^{k_1+\ldots+k_{n-1}+k_n+n} \omega_1 \otimes \ldots \otimes \omega_{n-1}\varepsilon_1(\omega_n).$$

The bar construction on $\Omega^*(X)$, $B^{\bullet}_{x_2,x_1} = (B^{\bullet}_{x_2,x_1}(\Omega^*(X)), d')$ is the total complex of the bicomplex $\oplus_{n=0}^{\infty} B^{-n}$ equipped with the differential $d' := \partial' + \delta'$.

2.3. Let $x_1, x_2, x_3 \in X$. We recall that there are chain maps

$$\mathcal{M}_{x_3x_2x_1} : B^{\bullet}_{x_3x_1} \to B^{\bullet}_{x_3x_2} \otimes B^{\bullet}_{x_2x_1}$$

given by

$$\mathcal{M}(\omega_1 \otimes \ldots \otimes \omega_n) = \sum_{i=0}^{n} (\omega_1 \otimes \ldots \omega_i) \otimes (\omega_{i+1} \otimes \ldots \omega_n)$$

and

$$\iota_{x_2x_1} : B^{\bullet}_{x_2x_1} \to B^{\bullet}_{x_1x_2}$$

given by

$$\iota_{x_2x_1}(\omega_1 \otimes \ldots \otimes \omega_n) := (-1)^{c(n)+1+(\frac{1}{2}+\frac{(-1)^n}{2})(k_1+\ldots k_n)+\sum_{i<j} k_ik_j} \omega_n \otimes \ldots \otimes \omega_1$$

where $c(n) = 0$ if $n \equiv 1, 2 \bmod 4$ and $c(n) = 1$ if $n \equiv 3, 4 \bmod 4$. Observe that $c(n) + 1 + (\frac{1}{2} + \frac{(-1)^n}{2})(k_1 + \ldots k_n) + \sum_{i<j} k_ik_j = \frac{n(n+1)}{2} + (n+1)(k_1 + \ldots k_n) + \sum_{i<j} k_ik_j$.

A (p, q)- shuffle is a permutation σ of $(1, 2, \ldots, p+q)$ such that $\sigma^{-1}(1) < \sigma^{-1}(2) < \ldots < \sigma^{-1}(p)$ and $\sigma^{-1}(p+1) < \sigma^{-1}(p+2) < \ldots < \sigma^{-1}(p+q)$.

We define the shuffle product $*$ as

$$(\omega_1 \otimes \ldots \otimes \omega_p) * (\omega_{p+1} \otimes \ldots \otimes \omega_{p+q}) = \sum_{(p,q)\text{ shuffles }\sigma} (-1)^{A_\sigma} (\omega_{\sigma(1)} \otimes \omega_{\sigma(2)} \otimes \ldots \otimes \omega_{\sigma(p+q)})$$

where $A_\sigma = A_\sigma(\omega_1, \ldots, \omega_{p+q}) = \sum_{i<j \text{ and } \sigma(i)>\sigma(j)} (k_{\sigma(i)} - 1)(k_{\sigma(j)} - 1)$.

The shuffle product

$$* : B^\bullet_{xy} \otimes B^\bullet_{xy} \to B^\bullet_{xy}$$

is a chain map. It is associative and $a * b = (-1)^{dega \cdot degb} b * a$.

Let $x = x_1 = x_2$. The inclusion $x \hookrightarrow X$ induces a chain map $e_x : B_{xx}(\Omega^*(X)) \to B_{xx}(\Omega^*(x))$.

Let C be a complex equal C in degree 0 and 0 in degree $n \neq 0$. The inclusion of C in $\Omega^0(X)$ induces

$$p_{x_2x_1} : B^\bullet_{x_2x_1}(C) \to B^\bullet_{x_2x_1}(\Omega^*(X)).$$

Proposition 2.3.1. *(see also [W1]). $B^\bullet_{x_2x_1}(\Omega^*(X))$ equipped with the shuffle product $*$ is a differential C-algebra. The maps $\mathcal{M}$, ι, e and p are homomorphims of differential C-algebras. We have*

$$(\mathcal{M}_{x_4x_3x_2} \otimes id_{B^\bullet_{x_2x_1}}) \circ \mathcal{M}_{x_4x_2x_1} = (id_{B^\bullet_{x_4x_3}} \otimes \mathcal{M}_{x_3x_2x_1}) \circ \mathcal{M}_{x_4x_3x_1},$$

$$* \circ (p_{x_2x_2} \circ e_{x_2}) \otimes id_{B^\bullet_{x_2x_1}} \circ \mathcal{M}_{x_2x_2x_1} = id_{B^\bullet_{x_2x_1}},$$

$$* \circ id_{B^\bullet_{x_2x_1}} \otimes (p_{x_1x_1} \circ e_{x_1}) \circ \mathcal{M}_{x_2x_1x_1} = id_{B^\bullet_{x_2x_1}},$$

$$* \circ (\iota_{xx} \otimes id_{B^\bullet_{xx}}) \circ \mathcal{M}_{xx} = p_{xx} \circ e_x,$$

$$* \circ (id_{B^\bullet_{xx}} \otimes \iota_{xx}) \circ \mathcal{M}_{xx} = p_{xx} \circ e_x.$$

Corollary 2.3.2. *(see also [W1]).* $\operatorname{Spec} H^0(B^\bullet_{xx})$ *is an affine pro-algebraic pro-unipotent group scheme over* C. $\operatorname{Spec} H^0(B^\bullet_{yx})$ *is a (right)* $\operatorname{Spec} H^0(B^\bullet_{xx})$*-torsor and a (left)* $\operatorname{Spec} H^0(B^\bullet_{yy})$*-torsor.*

Proof. It follows immediately from Proposition 2.3.1 that $\operatorname{Spec} H^0(B^\bullet_{xx})$ is an affine group scheme over C and that $\operatorname{Spec} H^0(B^\bullet_{yx})$ is a (right) $\operatorname{Spec} H^0(B^\bullet_{xx})$ - torsor and a (left) $\operatorname{Spec} H^0(B^\bullet_{yy})$ - torsor.

2.4. We define a map

$$\varepsilon : T_2 = Tot(\oplus_{n=0}^{\infty}\Omega^*(X)^{\otimes n}) \to B^\bullet(\Omega^*(X))$$

in such a way that ε on $\Omega^{k_1}(X)\otimes\ldots\otimes\Omega^{k_n}(X)$ is a multiplication by $(-1)^{c(n)+k_1+k_3+k_5+\cdots}$. The map ε is an isomorphism of chain complexes. We define the shuffle multiplication $*$, the coproduct $\mathcal{M}$ and the inverse ι on T_2 (and on T_1) setting :

$$\mathcal{M} \text{ on } T_2 : (\varepsilon\otimes\varepsilon)^{-1}\circ(\mathcal{M} \text{ on } B^\bullet)\circ\varepsilon$$

and similarly for ι and $*$. The maps e_x and p_{yx} can be defined in the same way.

One can also proceed in the following way. The projection on the constant cosimplicial space $p^\bullet : p^{\bullet -1}(y,x) \to (y,x)$ induces

$$p_{yx} : Tot(\oplus_{n=0}^{\infty}\Omega^*(y,x)) \to Tot(\oplus_{n=0}^{\infty}\Omega^*(y\times X^n\times x)).$$

Let (x) be a constant cosimplicial space equal one point x in each degree. The inclusion

$$(x) \hookrightarrow p^{\bullet -1}(x,x), \quad x \to (x,x,\ldots,x) \in x\times X^n\times x$$

induces

$$Tot(\oplus_{n=0}^{\infty}\Omega^*(x\times X^n\times x)) \to Tot(\oplus_{n=0}^{\infty}\Omega^*(x)).$$

2.5. Let G be a Lie group and let $\mathfrak{g}$ be a Lie algebra of G. Let $\pi : P \to X$ be a principal G-bundle equipped with the integrable connection given by a one form ω on P with values in $\mathfrak{g}$. Let us set

$$\mathcal{P} := \pi_*\mathcal{O}_P.$$

Then $\mathcal{P}$ is a vector bundle on X (an inductive limit of finite dimensional vector bundles) equipped with an integrable connection $\nabla_{\mathcal{P}}$ induced by the connection on P.

Let Ω^* be the De Rham complex of sheaves of smooth complex valued differential forms on X, let $\Omega^*(\mathcal{P})$ be the twisted De Rham complex and let $\Omega^*(\mathcal{P})(X)$ be the twisted de Rham complex of global sections.

For each point $x \in X$ we define an augmentation

$$\eta_x : \Omega^0(\mathcal{P})(X) \to \mathcal{O}[P_x],$$

where $\eta_x(f)$ is a restriction of $f \in \Omega^0(\mathcal{P})(X)$ to a fiber of $\mathcal{P}$ over x.

Let $y \in X$. Let us choose $y' \in \pi^{-1}(y)$. We define an augmentation

$$\varepsilon_{y'} : \Omega^0(\mathcal{P})(X) \to C$$

setting $\varepsilon_{y'}(f) := f(y')$.

Let $(y, x) \in X \times X$. Let

$$\partial : C \otimes (\Omega^*(\mathcal{P})(X))^{\otimes n} \otimes \mathcal{O}[P_x] \to C \otimes (\Omega^*(\mathcal{P})(X))^{\otimes n} \otimes \mathcal{O}[P_x]$$

be a differential of a tensor product of twisted De Rham complexes.

We define

$$\delta_i : C \otimes (\Omega^*(\mathcal{P})(X))^{\otimes n} \otimes \mathcal{O}[P_x] \to C \otimes (\Omega^*(\mathcal{P})(X))^{\otimes (n-1)} \otimes \mathcal{O}[P_x]$$

in the following way:

$$\delta_i(f_{n+1} \otimes \omega_n \otimes \ldots \otimes \omega_1 \otimes f_0) = f_{n+1} \otimes \omega_n \otimes \ldots \otimes \omega_{i+1} \wedge \omega_i \otimes \ldots \otimes \omega_1 \otimes f_0$$

for $i \neq 0, n$;

$$\delta_0(f_{n+1} \otimes \omega_n \otimes \ldots \otimes \omega_1 \otimes f_0) = f_{n+1} \otimes \omega_n \otimes \ldots \otimes \omega_2 \otimes \eta_x(\omega_1) f_0;$$

$$\delta_n(f_{n+1} \otimes \omega_n \otimes \ldots \otimes \omega_1 \otimes f_0) = f_{n+1} \varepsilon'_y(\omega_n) \otimes \omega_{n-1} \ldots \otimes \omega_1 \otimes f_0.$$

($\wedge$ is induced by the exterior product and the multiplication in $\mathcal{P}$.)

Let us set

$$\delta := \sum_{i=0}^{n}(-1)^i\delta_i : C \otimes (\Omega^*(\mathcal{P})(X))^{\otimes n} \otimes \mathcal{O}[P_x] \to C \otimes (\Omega^*(\mathcal{P})(X))^{\otimes(n-1)} \otimes \mathcal{O}[P_x].$$

We have $\partial^2 = \delta^2 = \partial\delta - \delta\partial = 0$. Therefore $\oplus_{n=0}^{\infty} C \otimes (\Omega^*(\mathcal{P})(X))^{\otimes n} \otimes \mathcal{O}[P_x]$ is a double complex and let T_{yx} be its total complex.

2.6. We define a bar construction

$$B^{\bullet}_{y,x}(C; \Omega^*(\mathcal{P})(X); \mathcal{O}[G])$$

being the same bigraded module as $T_{y,x}$ with the differential d' defined as in 2.2. We shall denote $B^{\bullet}_{y,x}(C; \Omega^*(\mathcal{P})(X); \mathcal{O}[G])$ shortly by $B^{\bullet}_{y,x}$.

The map $\varepsilon : T_{y,x} \to B^{\bullet}_{y,x}(C; \Omega^*(\mathcal{P})(X); \mathcal{O}[G])$ given by a multiplication by $(-1)^{c(n)+k_1+k_3+k_5+\ldots}$ on $(\Omega^{k_1}(X) \otimes \mathcal{O}[G]) \otimes \ldots \otimes (\Omega^{k_n}(X) \otimes \mathcal{O}[G]) \otimes \mathcal{O}[G]$ is an isomorphism of chain complexes.

The choice of $x' \in \pi^{-1}(x)$ identifies P_x - the fiber of P over x - with G, hence the fiber of $\mathcal{P}$ over x is identified with $\mathcal{O}[G]$.

We assume that the principal G - bundle $\pi : P \to X$ is trivial. Then $\Omega^*(\mathcal{P})(X) = \Omega^*(X) \otimes \mathcal{O}[G]$.

The action of G on P induces $\mathcal{P} \to \mathcal{P} \otimes \mathcal{O}[G]$, $f(p) \to f(pg)$. Let $x \in X$ be such that the fiber of $\mathcal{P}$ over x is $\mathcal{O}[G]$.

We define

$$\mathcal{M}_{yxx} : B^{\bullet}_{yx} \to B^{\bullet}_{yx} \otimes B_{xx}$$

by the formula

$$\mathcal{M}_{yxx}(f_0 \otimes \omega_1(p_1) \otimes \ldots \otimes \omega_1(p_1) \otimes f_{n+1}(g_{n+1})) =$$

$$\sum_{i=0}^{n}(f_0 \otimes \omega_1(p_1) \otimes \ldots \otimes \omega_i(p_i) \otimes 1) \otimes$$

$$(1 \otimes \omega_{i+1}(p'_{i+1}g_{i+1}) \otimes \ldots \otimes \omega_n(p'_n g_{i+1}) \otimes f_{n+1}(g'_{n+1}g_{i+1})).$$

We define

$$\iota : B^{\bullet}_{xx} \to B^{\bullet}_{xx}$$

by the formula

$$\iota(f_0 \otimes \omega_1(p_1) \otimes \ldots \otimes \omega_1(p_1) \otimes f_{n+1}(g_{n+1})) =$$

$$(-1)^{\frac{n(n+1)}{2}+(n+1)(k_1+\ldots+k_n)+\sum_{i<j} k_i k_j} f_0 \otimes \omega_n(p_n g_{n+1}^{-1}) \otimes \ldots \otimes \omega_1(p_1 g_{n+1}^{-1}) \otimes f_{n+1}(g_{n+1}^{-1}).$$

We define the shuffle product

$$* : B^\bullet_{yx} \otimes B^\bullet_{yx} \to B^\bullet_{yx}$$

by the formula

$$(f_0 \otimes \omega_1 \otimes \ldots \otimes \omega_p \otimes f_{p+1}) * (g_p \otimes \omega_{p+1} \otimes \ldots \otimes \omega_{p+q} \otimes g_{p+q+1}) =$$

$$\sum_{(p,q)-\text{shuffles}\,\sigma} (-1)^{A_\sigma} f_0 g_p \otimes \omega_{\sigma(1)} \otimes \ldots \omega_{\sigma(p+q)} \otimes f_{p+1} g_{p+q+1}.$$

The sign $(-1)^{A_\sigma}$ is defined as in 2.3.

The inclusion $x \hookrightarrow X$ induces

$$e_x : B^\bullet_{x,x}(C; \Omega^*(\mathcal{P})(X); \mathcal{O}[G]) \to B^\bullet_{x,x}(C; \mathcal{O}[G]; \mathcal{O}[G]).$$

$B^\bullet_{y,x}$ equipped with the shuffle product is a C - algebra, hence we have a morphism of C - algebras

$$p_{yx} : C \to B^\bullet_{y,x}.$$

Proposition 2.6.1. *$B^\bullet_{y,x}(C; \Omega^*(\mathcal{P})(X); \mathcal{O}[G])$ equipped with the shufle product $*$ is a differential C - algebra. The maps $\mathcal{M}$, ι , e and p are homomorphims of differential C-algebras. We have*

$$(\mathcal{M}_{yxx} \otimes id_{B^\bullet_{xx}}) \circ \mathcal{M}_{yxx} = (id_{B^\bullet_{yx}} \otimes \mathcal{M}_{xxx}) \circ \mathcal{M}_{yxx},$$

$$* \circ (p_{xx} \circ e_x) \otimes id_{B^\bullet_{xx}} \circ \mathcal{M}_{xxx} = id_{B^\bullet_{xx}},$$

$$* \circ id_{B^\bullet_{xx}} \otimes (p_{xx} \circ e_x) \circ \mathcal{M}_{xxx} = id_{B^\bullet_{xx}},$$

$$* \circ (\iota_{xx} \otimes id_{B^\bullet_{xx}}) \circ \mathcal{M}_{xx} = p_{xx} \circ e_x,$$

$$* \circ (id_{B^\bullet_{xx}} \otimes \iota_{xx}) \circ \mathcal{M}_{xx} = p_{xx} \circ e_x.$$

Proof. One checks the formulas by calculations.

Passing to cohomology the maps $\mathcal{M}$, ι and e induce

$$\mathcal{M}_{yx} : H^0(T_{yx}) \to H^0(T_{yx}) \otimes H^0(T_{xx}),$$

$$\iota_{yx} : H^0(T_{yx}) \to H^0(T_{xy}) \text{ and } e_x : H^0(T_{xx}) \to C.$$

$H^0(T_{yx})$ equipped with the shufle product is a C - algebra. Hence we have a homomorphism of C - algebras

$$p_{yx} : C \to H^0(T_{yx}).$$

Proposition 2.5.2. *$H^0(T_{yx})$ equipped with the shuffle product is a C-algebra. The maps $\mathcal{M}_{yx}$, ι_{yx} and e_x are homomorphims of differential C-algebras. We have*

$$(\mathcal{M}_{yx} \otimes id_{H^0(T_{xx})}) \circ \mathcal{M}_{yx} = (id_{H^0(T_{yx})} \otimes \mathcal{M}_{xx}) \circ \mathcal{M}_{yx},$$

$$* \circ (p_{xx} \circ e_x) \otimes id_{H^0(T_{xx})} \circ \mathcal{M}_{xx} = id_{H^0(T_{xx})},$$

$$* \circ id_{H^0(T_{yx})} \otimes (p_{xx} \circ e_x) \circ \mathcal{M}_{yx} = id_{H^0(T_{xx})},$$

$$* \circ (\iota_{xx} \otimes id_{H^0(T_{xx})}) \circ \mathcal{M}_{xx} = p_{xx} \circ e_x,$$

$$* \circ (id_{H^0(T_{xx})} \otimes \iota_{xx}) \circ \mathcal{M}_{xx} = p_{xx} \circ e_x.$$

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2.6. We define a bar construction

$\mathrm{B}^\bullet_{y,x}(\mathcal{O}[P_y] \otimes \Omega^*(X) \otimes \mathcal{O}[G]; \mathcal{O}[G]) being the same bigraded module as \mathrm{T}_{y,x}$ with the differential d' defined as in 2.2. The map $\varepsilon : T_{y,x} \to B^\bullet_{y,x}(\Omega^*(X) \otimes \mathcal{O}[G]; \mathcal{O}[G])$ given by a

multiplication by $(-1)^{c(n)+k_1+k_3+k_5+\dots}$ on $\Omega^{k_1}(X)\otimes\mathcal{O}[G])\otimes\dots\otimes(\Omega^{k_n}(X)\otimes\mathcal{O}[G])\otimes\mathcal{O}[G])$ is an isomorphism of chain complexes.

The comultiplication $\mathcal{M}$, the inverse ι and the product $*$ are given on $B^\bullet(\Omega^*(X)\otimes\mathcal{O}[G];\mathcal{O}[G])$ with the sign convention as in 2.2.

2.7. We have the following analogues of 2.3.1 and 2.3.2.

Proposition 2.7.1. *$T_{x_2x_1}(\Omega^*(X))$ equipped with the shuffle product $*$ is a differential C-algebra. The maps $\mathcal{M}$, ι, e and p are homomorphims of differential C-algebras. We have*

$$(\mathcal{M}_{x_4x_3x_2}\otimes id_{T_{x_2x_1}})\circ\mathcal{M}_{x_4x_2x_1}=(id_{T_{x_4x_3}}\otimes\mathcal{M}_{x_3x_2x_1})\circ\mathcal{M}_{x_4x_3x_1},$$

$$*\circ(p_{x_2x_2}\circ e_{x_2})\otimes id_{T_{x_2x_1}}\circ\mathcal{M}_{x_2x_2x_1}=id_{T_{x_2x_1}},$$

$$*\circ id_{T_{x_2x_1}}\otimes(p_{x_1x_1}\circ e_{x_1})\circ\mathcal{M}_{x_2x_1x_1}=id_{T_{x_2x_1}},$$

$$*\circ(\iota_{xx}\otimes idT_{xx})\circ\mathcal{M}_{xx}=p_{xx}\circ e_x,$$

$$*\circ(id_{T_{xx}}\otimes\iota_{xx})\circ\mathcal{M}_{xx}=p_{xx}\circ e_x.$$

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Corollary 2.5.3. *We have*

- *i).* Spec $H^0(T_{xx})$ *is an afine pro-algebraic group scheme over C.*
- *ii).* Spec $H^0(T_{yx})$ *is a (right)* Spec $H^0(T_{xx})$*-torsor.*
- *iii). The group scheme* Spec $H^0(T_{x,x})$ *is an extention of* Spec $\mathcal{O}[G]$ *by an affine pro - unipotent group over C.*

Proof. It follows from Proposition 2.5.2 that Spec $H^0(T_{xx})$ is a group scheme over C and that Spec $H^0(T_{yx})$ is a Spec $H^0(T_{xx})$ - torsor.

First we show that there is a surjective morphism of group schemes

$$\operatorname{Spec} H^0(T_{xx}) \to \operatorname{Spec} \mathcal{O}[G].$$

Observe that $T_{xx}^{00} = \mathcal{O}[G] \otimes \mathcal{O}[G]$, $T_{xx}^{10} = \mathcal{O}[G] \otimes \mathcal{O}[G]$ and $T_{xx}^{01} = \mathcal{O}[G] \otimes \mathcal{O}_X(\mathcal{P})(X) \otimes \mathcal{O}[G]$. The differential

$$\delta : T_{xx}^{01} \to T_{xx}^{00}$$

is given by $\delta(f \otimes g \otimes h) = (f\eta_x(g) \otimes h - f \otimes \eta_x(g)h)$, where $f, h \in \mathcal{O}[G]$ and $g \in \mathcal{O}_X(\mathcal{P})$. Therefore $\mathcal{O}[G]$ is a Hopf subalgebra of $H^0(T_{xx})$ and we have an exact sequence

$$1 \to U \to \operatorname{Spec} H^0(T_{xx}) \xrightarrow{\varphi} \operatorname{Spec} \mathcal{O}[G] \to 1,$$

where $U := \ker \varphi$.

Let us set

$$F_n T_{xx} := \oplus_{i<n+1} T_{xx}^{*i}.$$

Let $F_n H^0(T_{xx})$ be a subalgebra of $H^0(T_{xx})$ generated by $im(H^0(F_n T_{xx}) \to H^0(T_{xx}))$. Observe that $F_n H^0(T_{xx})$ is a finitely generated Hopf subalgebra of $H^0(T_{xx})$. This implies that $\operatorname{Spec} H^0(T_{xx})$ is an affine pro-algebraic group scheme.

Let us set

$$G^{(n)} := \operatorname{Spec} F_n H^0(T_{xx}).$$

The inclusion

$$F_{n-1} H^0(T_{xx}) \to F_n H^0(T_{xx})$$

induces a surjective morphism of group scheme

$$\varphi_n : G^{(n)} \to G^{(n-1)}.$$

Observe that $G^{(0)} = G$ and $\operatorname{Spec} H^0(T_{xx}) = \varprojlim_n G(n)$.

Let us set

$$U^{(n)} := \ker(G^{(n)} \to G^{(0)}).$$

We have $U = \varprojlim_n U^{(n)}$. We shall show that for any n, the group $A_n := \ker(\varphi_n : U^{(n)} \to U^{(n-1)})$ is abelian. This implies that U is pro - unipotent.

Observe that $A_1 = U^{(1)}$. We have an exact sequence of groups

$$1 \to U^{(1)} \to G^{(1)} \to G^{(0)} \to 1$$

and the corresponding sequence of coordinate rings

$$\mathcal{O}[G] \hookrightarrow F_1 H^0(T_{xx}) \to F_1 H^0(T_{xx})/I(U^{(1)}).$$

The kernel of the augmentation homomorphism $\varepsilon : \mathcal{O}[G] \to C$ $(f \to f(e))$ is contained in $I(U^{(1)})$. We have $\mathcal{M}(1 \otimes w \otimes 1) = (1 \otimes w \otimes 1) \otimes (1 \otimes 1) + \sum_i (1 \otimes f_i) \otimes (1 \otimes w^i \otimes 1)$, where $m : \Omega^1_X(\mathcal{P})(X) \to \mathcal{O}[G] \otimes \Omega^1_X(\mathcal{P})(X)$ is induced by the action of G on P and $m(w) = \sum_i f_i \otimes w_i$. Observe that $1 \otimes f_i \equiv 1 \otimes f_i(e) \bmod I(U^{(1)})$. Hence $\mathcal{M}(1 \otimes w \otimes 1) = (1 \otimes w \otimes 1) \otimes (1 \otimes 1) + (1 \otimes 1) \otimes (1 \otimes w \otimes 1) \bmod I(U^{(1)})$. This implies that $U^{(1)}$ is abelian.

To show that A_n is abelian one observes that $A_n := \ker(G^{(n)} \to G^{(n-1)})$ and one repeats the proof given above for A_1.

§3. Universal variation of mixed Hodge structures.

3.0. Let X be a topological space. The inclusion of simplicial sets

$$\partial\Delta[1] \hookrightarrow \Delta[1]$$

induces a morphism of cosimplicial spaces

$$p^\bullet : X^{\Delta[1]} \to X^{\partial\Delta[1]}.$$

Observe that $X^{\partial\Delta[1]}$ is a constant cosimplicial space equal $X \times X$ in each degree. Let x and y be two points of X. Then constant cosimplicial spaces $(X \times x)^\bullet$, $(y \times X)^\bullet$ and $(y \times x)^\bullet$ are cosimplicial subspaces of $X^{\partial\Delta[1]}$. Let us set

$$(X; y)^{(\Delta[1],1)} := p^{\bullet -1}((y \times X)^\bullet), \;\; (X; x)^{(\Delta[1];0)} := p^{\bullet -1}((X \times x)^\bullet),$$

$$(X; x, y)^{(\Delta[1];0,1)} := p^{\bullet -1}((y \times x)^\bullet) \text{ and } (X; x)^{(S^1;1)} := p^{\bullet -1}((x \times x)^\bullet).$$

We shall use also the notation $(y \times X)^\bullet = (X; y)^{(\partial\Delta[1],1)}$. We recall that the cosimplicial space $X^{\Delta[1]}$ is given by

$$X \times X \overset{\delta^0,\, \delta^1}{\longrightarrow} X \times X \times X \overset{\delta^0,\, \delta^1,\, \delta^2}{\longrightarrow} X \times X^2 \times X \ldots,$$

where

$$\delta^i : X \times X^n \times X \to X \times X^{n+1} \times X, \;\; \delta^i(x_{n+1}, x_n, \ldots, x_1, x_0) = (x_{n+1}, x_n, \ldots, x_i, x_i, \ldots, x_1, x_0).$$

3.1. We shall consider parallelly two cases:

i) X is a smooth holomorphic variety and Ω^*_X is a holomorphic De Rham complex. G is an affine complex reductive algebraic group and $\pi : P \to X$ is a holomorphic principal G-bundle equipped with a holomorphic integrable connection.

ii) X is a smooth quasi-projective scheme of finite type over a field k of characteristic zero, G is an affine reductive algebraic group over $\operatorname{Spec} k$ and $\pi : P \to X$ is a principal G-bundle locally trivial in Zariski topology (resp. etale topology, resp. flat topology) of X. Ω^*_X is

the algebraic De Rham complex on X_{Zar} (resp. X_{et}, resp. X_{flat}) - X equipped with the corresponding topology.

Let $\mathcal{P} := \pi_* \mathcal{O}_P$. This is an inductive limit of finite dimensional vector bundles on X in a corresponding topology. The sheaf $\mathcal{P}$ is a sheaf of $\mathcal{O}_X$ -algebras. The connection on the principal G-bundle $\pi : P \to X$ induces a multiplicative integrable connection ∇ on $\mathcal{P}$ i.e.

$$\nabla(a \cdot b) = \nabla(a) \cdot b + a \cdot \nabla(b),$$

where a and b are sections of $\mathcal{P}$. Hence the tensor products of sheaves $\mathcal{P}^{\otimes(n+2)}$ on $X \times X^n \times X$ define a sheaf $\mathcal{P}^{\otimes(\bullet+2)}$ on a cosimplicial space $X^{\Delta[1]}$. The twisted De Rham complexes $\Omega^*_{X^{n+2}}(\mathcal{P}^{\otimes(n+2)})$ define a twisted De Rham complex $\Omega^*_{X^{\Delta[1]}}(\mathcal{P}^{\otimes(\bullet+2)})$ on $X^{\Delta[1]}$. The relative twisted De Rham complexes $\Omega^*_{(X\times X^n\times X)/(X\times X)}(\mathcal{P}^{\otimes(n+2)})$ define a relative twisted De Rham complex

$$\Omega^* := \Omega^*_{X^{\Delta[1]}/(X\times X)^\bullet}(\mathcal{P}^{\otimes(\bullet+2)})$$

on a cosimplicial space $X^{\Delta[1]}$.

We shall study the cohomology sheaves $H^i(tR(p^\bullet)_*\Omega^*)$ of $\mathcal{O}_{X\times X}$-modules, where $p^\bullet : X^{\Delta[1]} \to X^{\partial\Delta[1]}$.

The fiber of $H^i(tRp^\bullet_*\Omega^*)$ over $(y,x) \in X \times X$ can be describe in the following way. The restriction of $\Omega_{X^{n+2}}(\mathcal{P}^{\otimes(n+2)})$ to $y \times X^n \times x$ is $\mathcal{O}[P_y] \otimes \Omega_{y\times X^n\times x}(\mathcal{P}^{\otimes n}) \otimes \mathcal{O}[P_x]$. Hence the fiber of $H^i(tRp^\bullet_*\Omega^*)$ over (y,x) is $H^i(tR\Gamma\Omega^*_{(X;y,x)^{(\Delta[1];y,x)}}(\mathcal{O}[P_y] \otimes \mathcal{P}^{\otimes\bullet} \otimes \mathcal{O}[P_x])$.

3.2. We assume that $\mathcal{O}[G]$ carries a Hodge structure of weight 0 and that the structure morphisms and the multiplication are morphisms of Hopf structures. We assume that $\mathcal{P}$ equipped with the connection deduced from the connection on $\pi : P \to X$ carries a variation of Hodge structures of weight 0 compatible with multiplication in $\mathcal{P}$ and with the action of $\mathcal{O}[G]$ on $\mathcal{P}$.

3.2.1. Our aim is to show that the cohomology sheaves $H^i(tRp^\bullet_*\Omega^*)$ carry a variation of mixed Hodge structures over X. Next we shall show that $H^0(tRp^\bullet_*\Omega^*)$ is the sheaf of $\mathcal{O}_{X\times X}$-algebras and that the variation of mixed Hodge structures is compatible with multiplication. Let $\mathcal{H}$ be a fiber of $H^0(tRp^\bullet_*\Omega^*)$ over (x,x). We shall show that $\mathcal{H}$ is

a Hopf algebra and that the Hopf algebra structure and the action are compatible with mixed Hodge structures.

3.3. We assume that X is a smooth projective variety over C.

Let $\mathcal{V}$ be a variation of Hodge structures of weight k over X and let $\{\mathcal{F}^p\mathcal{V}\}_p$ be a Hodge filtration of $\mathcal{V}$. Let us equip the complex $\Omega^*_X(\mathcal{V})$ with a filtration $\{F^p\}_p$ in the following way :

$$F^p(\Omega^r_X(\mathcal{V})) := \Omega^r_X(\mathcal{F}^{p-r}(\mathcal{V})).$$

Let $\mathcal{V}_0$ be a rational lattice in $\mathcal{V}$. It is a local system of vector spaces over Q. We have

$$ker(\nabla : \mathcal{V} \to \Omega^1_X(\mathcal{V})) \approx \mathcal{V}_0 \otimes C.$$

Hence there is a quasi-isomorphism $\alpha : \mathcal{V}_0 \otimes C \to \Omega^*_X(\mathcal{V})$. It follows from [Z] Theorem 2.9 that

$$(\mathcal{V}_0, (\Omega^*_X(\mathcal{V}), F), \alpha : \mathcal{V}_0 \otimes C \to \Omega^*_X(\mathcal{V}))$$

is a cohomological Hodge complex of weight k over X.

3.3.1. For each n, $\mathcal{P}^{\otimes n}$ is a variation of Hodge structures of weight zero on $y \times X^n \times x$. Hence

$$(\mathcal{O}[P_y]_0 \otimes \mathcal{P}_0^{\otimes n} \otimes \mathcal{O}[P_x]_0, (\Omega^*_{y\times X^n\times x}(\mathcal{O}[P_y] \otimes \mathcal{P}^{\otimes n} \otimes \mathcal{O}[P_x]), F),$$

$$\alpha_n : \mathcal{O}[P_y]_0 \otimes \mathcal{P}_0^{\otimes n} \otimes \mathcal{O}[P_x]_0 \otimes C \to \Omega^*_{y\times X^n\times x}(\mathcal{O}[P_y] \otimes \mathcal{P}^{\otimes n} \otimes \mathcal{O}[P_x]))$$

is a cohomological Hodge complex of weight zero on $y \times X^n \times x$.

3.3.2. The complexes $\Omega^*_{y\times X^n\times x}(\mathcal{O}[P_y] \otimes \mathcal{P}^{\otimes n} \otimes \mathcal{O}[P_x])$ equipped with the filtration F form a complex $\Omega^*_{(X;x,y)^{(\Delta[1];0,1)}}(\mathcal{O}[P_y] \otimes \mathcal{P}^\bullet \otimes \mathcal{O}[P_x])$ equipped with the filtration F on the cosimplicial space $(X;x,y)^{(\Delta[1];0,1)}$. The sheaves $\mathcal{O}[P_y]_0 \otimes \mathcal{P}_0^{\otimes n} \otimes \mathcal{O}[P_x]_0$ form a sheaf $\mathcal{O}[P_y]_0 \otimes \mathcal{P}_0^{\otimes \bullet} \otimes \mathcal{O}[P_x]_0$ on $(X;x,y)^{(\Delta[1];0,1)}$. The morphisms α_n induce a quasi - isomorphism

$$\alpha : \mathcal{O}[P_y]_0 \otimes \mathcal{P}_0^{\otimes \bullet} \otimes \mathcal{O}[P_x]_0 \otimes C \overset{\approx}{\to} \Omega^*_{(X;x,y)^{(\Delta[1];0,1)}}(\mathcal{O}[P_y]_0 \otimes \mathcal{P}^{\otimes \bullet} \otimes \mathcal{O}[P_x]_0).$$

Hence the triple

$$\big(\mathcal{O}[P_y]_0 \otimes \mathcal{P}_0^{\otimes \bullet} \otimes \mathcal{O}[P_x]_0, (\Omega^*_{(X;x,y)^{(\Delta[1];0,1)}}(\mathcal{O}[P_y] \otimes \mathcal{P}^{\otimes \bullet} \otimes \mathcal{O}[P_x]), F),$$

$$\alpha : \mathcal{O}[P_y]_0 \otimes \mathcal{P}_0^{\otimes\bullet} \otimes \mathcal{O}[P_x]_0 \otimes C \to \Omega^*_{(X;x,y)^{(\Delta[1];0,1)}}(\mathcal{O}[P_y] \otimes \mathcal{P}^{\otimes\bullet} \otimes \mathcal{O}[P_x])\big)$$

is a cohomological Hodge complex on a cosimplicial space $(X;x,y)^{(\Delta[1];0,1)}$.

3.3.3. Let $K = (K^{i,j})$ be a simplicial complex (j is a simplicial degree). We define a filtration $\partial(W,R)$ on the total complex tK in the following way :

$$\partial(W,R)_n(tK) := \oplus_{j<n+1} K^{i,j}.$$

Appying the functor $R\Gamma^\bullet$ ($R\Gamma$ functorially in each degree) we get the following result.

Theorem 3.3.4. *The triple*

$$\big((tR\Gamma^\bullet(\mathcal{O}[P_y]_0 \otimes \mathcal{P}_0^{\otimes\bullet} \otimes \mathcal{O}[P_x]_0); \partial(W,R)\big), \big(tR\Gamma^\bullet\Omega^*_{(X;x,y)^{(\Delta[1];0,1)}}(\mathcal{O}[P_y] \otimes \mathcal{P}^{\otimes\bullet} \otimes \mathcal{O}[P_x]), \partial(W,R), F\big),$$

$$tR\Gamma^\bullet\alpha := (tR\Gamma^\bullet(\mathcal{O}[P_y]_0 \otimes \mathcal{P}_0^{\otimes\bullet} \otimes \mathcal{O}[P_x]_0) \otimes C, \partial(W,R)) \to$$

$$tR\Gamma^\bullet\Omega^*_{(X;x,y)^{(\Delta[1];0,1)}}(\mathcal{O}[P_y] \otimes \mathcal{P}^{\otimes\bullet} \otimes \mathcal{O}[P_x]), \partial(W,R))$$

is a mixed Hodge complex.

Corollary 3.3.5. *The cohomology groups* $H^i(tRp^\bullet_*\Omega^*_{(X;x,y)^{(\Delta[1];0,1)}}(\mathcal{O}[P_y] \otimes \mathcal{P}^{\otimes\bullet} \otimes \mathcal{O}[P_x]))$ *are equipped with mixed Hodge structures.*

3.4. We give a relative version of the construction from 3.3.

The Hodge filtration on $\mathcal{P}$ induces the Hodge filtration $\{F^p\}$ on Ω^*. Observe also that the complex Ω^* is a resolution of $(p^{\bullet-1})\mathcal{O}_{X\times X} \otimes_Q \mathcal{P}_0^{\otimes(\bullet+2)}$.

Proposition 3.4.1. *The triple*

$$(\mathcal{P}_0^{\otimes(\bullet+2)}; (\Omega^*, F); \alpha^\bullet : (p^{\bullet-1})\mathcal{O}_{X\times X} \otimes_Q \mathcal{P}_0^{\otimes(\bullet+2)} \to \Omega^*)$$

is a relative w.r.t. $p^\bullet : X^{\Delta[1]} \to X^{\partial\Delta[1]}$ *cohomological Hodge complex on* $X^{\Delta[1]}$.

Proposition 3.4.2. *The triple*

$$((tRp^\bullet_*\mathcal{P}_0^{\otimes(\bullet+2)}, \partial(W,R)), (tRp^\bullet_*\Omega^*, \partial(W,R), F), \alpha := tRp^\bullet_*\alpha^\bullet : tRp^\bullet_*\mathcal{P}_0^{\otimes(\bullet+2)} \otimes_Q \mathcal{O}_{X\times X} \to tRp^\bullet_*\Omega^*)$$

is a mixed Hodge complex on $X \times X$.

We denote by W the filtration induced by $\partial(W,R)$ on cohomology groups.

Theorem 3.4.3. *The triple*

$$((H^i(tRp_*^\bullet \mathcal{P}_0^{\otimes(\bullet+2)}); W), (H^i(tRp_*^\bullet \Omega^*), W, F),$$

$$\alpha : H^i(tRp_*^\bullet \mathcal{P}_0^{\otimes(\bullet+2)}) \otimes_Q \mathcal{O}_{X\times X}, W) \to (H^i(tRp_*^\bullet \Omega^*), W))$$

is a variation of mixed Hodge structures over $X \times X$.

Proof. One need only to construct an integrable connection $\nabla : H^i(tRp_*^\bullet \Omega^*) \to \Omega^1_{X\times X} \otimes_{\mathcal{O}_{X\times X}} H^i(tRp_*^\bullet \Omega^*)$, which has the required properties. This is done in the next sections.

3.6. The Gauss-Manin connection.

Let us set:

$$G_n^i(\Omega^*_{X^{n+2}}(\mathcal{P}^{\otimes(n+2)})) :=$$

$$\text{image}(\Omega^{*-i}_{X^{n+2}}(\mathcal{P}^{\otimes(n+2)}) \otimes_{\mathcal{O}_{X\times X}} (p^n)^* \Omega^i_{X\times X} \to \Omega^*_{X^{n+2}}(\mathcal{P}^{\otimes(n+2)})).$$

The filtrations $\{G_n^i\}_i$ of $\Omega^*_{X^{n+2}}(\mathcal{P}^{\otimes(n+2)})$ define a filtration $\{G^i\}_i$ of $\Omega^*_{X^{\Delta[1]}}(\mathcal{P}^{\otimes(\bullet+2)})$. We have

$$gr_G^i(\Omega^*_{X^{\Delta[1]}}(\mathcal{P}^{\otimes(\bullet+2)})) = \Omega^{*-i} \otimes_{\mathcal{O}_{X^{\Delta[1]}}} (p^\bullet)^* \Omega^i_{(X\times X)^\bullet}.$$

Consider the functor $t \circ R(p^\bullet)_*$ from the category of complexes of abelian sheaves on $X^{\Delta[1]}$ to the category of complexes of abelian sheaves on $X \times X$. We apply the spectral sequence of a filtered object to $\Omega^*_{X^{\Delta[1]}}(\mathcal{P}^{\otimes(\bullet+2)})$ and we get a spectral sequence converging to $H^i(tR(p^\bullet)_* \Omega^*_{X^{\Delta[1]}}(\mathcal{P}^{\otimes(\bullet+2)}))$ such that $E_1^{p,q} = \Omega^p_{X\times X} \otimes_{\mathcal{O}_{X\times X}} H^q(tR(p^\bullet)_* \Omega^*)$. Let $d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$ be the differential on E_1 -terms.

Theorem 3.6.1. *Let X be a smooth scheme of finite type over a field k of characteristic zero. We have:*

i). The differential

$$d_1^{0,q} : H^q(tRp_*^\bullet \Omega^*) \to \Omega^1_{X\times X} \otimes_{\mathcal{O}_{X\times X}} H^q(tRp_*^\bullet \Omega^*)$$

is an integrable connection on $H^q(tRp_*^\bullet \Omega^*)$. If f and g are sections of $H^q(tRp_*^\bullet \Omega^*)$ and $H^{q'}(tRp_*^\bullet \Omega^*)$ respectively then

$$d_1^{0,q+q'}(f \cdot g) = d_1^{0,q}(f) \cdot g + (-1)^q f \cdot d_1^{0,q'}(g)$$

ii). Let F^i be the Hodge filtration of $H^q(tRp_*^\bullet\Omega^*)$. Then we have

$$d_1^{0,q}(F^iH^q(tRp_*^\bullet\Omega^*)) \subset \Omega^1_{X\times X}\otimes_{\mathcal{O}_{X\times X}} F^{i-1}H^q(tRp_*^\bullet\Omega^*).$$

iii). Assume that X is also projective. Then we have

$$d_1^{0,q}(W_nH^q(tRp_*^\bullet\Omega^*)) \subset \Omega^1_{X\times X}\otimes_{\mathcal{O}_{X\times X}} W_nH^q(tRp_*^\bullet\Omega^*).$$

iv). The multiplication and the action of $\mathcal{H}$ on $H^0(tRp_*^\bullet\Omega^*)$; $H^0(tRp_*^\bullet\Omega^*)\to\mathcal{H}\otimes H^0(tRp_*^\bullet\Omega^*)$ are flat.

v). The sheaves $H^q(tRp_*^\bullet\Omega^*)$ are direct limits of vector bundles on $X\times X$.

vi). Let $K:k$ be an extension of fields. Then the cohomology sheaves and the connection $d_1^{0,q}$ for $X_K = X\times_{\operatorname{Spec} k}\operatorname{Spec} K$ are obtained from the ones for X by applying the functor $-\otimes_k K$.

vii). Let X be a smooth holomorphic variety and let $\Pi^{\otimes(\bullet+2)} := \ker(\nabla : \mathcal{P}^{\otimes(\bullet+2)}\to\Omega^*_{X^{\Delta[1]}}(\mathcal{P}^{\otimes(\bullet+2)})$. Then $H^q(tRp_*^\bullet\Pi^{\otimes(\bullet+2)})$ is the sheaf of flat sections of the connection $d_1^{0,q}$ on $H^q(tRp_*^\bullet\Omega^*)$.

Proof. The complex $tRp_*^\bullet\Omega^*_{X^{\Delta[1]}}(\mathcal{P}^{\otimes(\bullet+2)})$ is equipped with shuffle product. The filtration $\{G^i\}_i$ is compatible with the multiplicative structure. Hence the spectral sequence is multiplicative and

$$d_1^{0,q+q'}(f\cdot g) = d_1^{0,q}(f)\cdot g + (-1)^q f\cdot d_1^{0,q'}(g)$$

if $f\in H^q(tRp_*^\bullet\Omega^*)$ and $g\in H^{q'}(tRp_*^\bullet\Omega^*)$.

The differential d_1 has a bidegree $(1,0)$ therefore for every q we get a complex

$$(*_q)\ \ 0\to\mathcal{O}_{X\times X}\otimes_{\mathcal{O}_{X\times X}} H^q(tRp_*^\bullet\Omega^*)\to\Omega^1_{y\times X}\otimes_{\mathcal{O}_{y\times X}} H^q(tRp_*^\bullet\Omega^*)\to$$

$$\Omega^2_{X\times X}\otimes_{\mathcal{O}_{X\times X}} H^q(tRp_*^\bullet\Omega^*)\to\dots.$$

Consider the diagram of cosimplicial spaces

$$\begin{array}{ccc} X^{\Delta[1]} & \xrightarrow{p^\bullet} & X^{\partial\Delta[1]} = (X\times X)^\bullet \\ \downarrow p^\bullet & & \downarrow id \\ X^{\partial\Delta[1]} & \xrightarrow{id} & X^{\partial\Delta[1]} \end{array}$$

The horizontal map $p^\bullet$ induces a map of complexes of sheaves

$$\Omega^*_{X^{\partial\Delta[1]}} \to (p^\bullet)_*\Omega^*_{X^{\Delta[1]}}(\mathcal{P}^{\otimes(\bullet+2)})$$

given by

$$\Omega^*_{X\times X}(\mathcal{U}) \ni \omega \to (p^n)^*\omega \otimes 1 \in \Omega^*_{X^{n+2}}(\mathcal{P}^{\otimes(n+2)}).$$

We repeat the construction of the spectral sequence for the vertical morphism id. The pair of horizontal morphisms $(p^\bullet, id)$ induces a map of corresponding spectral sequences. On $E_1^{*,0}$ - terms we get a morphism of $E_1^{*,0}(id) = \Omega^*_{X\times X}$ into the complex $E_1^{*,0}(p^\bullet) = (*_0)$. Let $f \in \mathcal{O}_{X\times X}$ and $s \in H^q(tRp^\bullet_*\Omega^*)$. It follows from the above observations and multiplicative properties of the spectral sequence that $d_1^{0,q}(f\cdot s) = df\cdot s + f\cdot d_1^{0,q}(s)$. Hence $d_1^{0,q}$ is a connection. Moreover $d_1^{p,q}(\omega\cdot s) = d\omega\cdot s + (-1)^p\omega\cdot d_1^{0,q}(s)$ if $\omega \in \Omega^p_{X\times X}$. Hence $d_1^{p,q}$ is deduced formally, as a differential of a twisted De Rham complex from the connection $d_1^{0,q}$. Hence the fact $d_1^{1,q}\circ d_1^{0,q} = 0$ implies that the connection is integrable.

The connection $d_1^{0,q}$ is the connecting homomorphism of the exact sequence

$$(**)\ 0 \to \Omega^{*-1}\otimes_{\mathcal{O}_{X^{\Delta[1]}}} (p^\bullet)^*\Omega^1_{X^{\partial\Delta[1]}} \to G^0/G^2 \to$$

$$\Omega^*\otimes_{\mathcal{O}_{X^{\Delta[1]}}} (p^\bullet)^*\mathcal{O}_{X^{\partial\Delta[1]}} \to 0.$$

Observe that an exact sequence

$$0 \to F^{n-1}\Omega^{*-1}\otimes_{\mathcal{O}_{X^{\Delta[1]}}} (p^\bullet)^*\Omega^1_{X^{\partial\Delta[1]}} \to F^n(G^0/G^2) \to$$

$$F^n\Omega^*\otimes_{\mathcal{O}_{X^{\Delta[1]}}} (p^\bullet)^*\mathcal{O}_{X^{\partial\Delta[1]}} \to 0.$$

is a sub sequence of $(**)$. This implies

$$d_1^{0q}(F^nH^q(tRp^\bullet_*\Omega^*)) \subset \Omega^1_{X\times X}\otimes_{\mathcal{O}_{X\times X}} F^{n-1}H^q(tRp^\bullet_*\Omega^*).$$

Let us consider the exact sequence $(**)$ on the n-truncation of the cosimplicial space $X^{\Delta[1]}$. Then it follows immediately that

$$d_1^{0,q}(W_nH^q(tRp^\bullet_*\Omega^*)) \subset \Omega^1_{X\times X}\otimes_{\mathcal{O}_{X\times X}} W_nH^q(tRp^\bullet_*\Omega^*)$$

if X is a smooth projective algebraic variety over $\operatorname{Spec} k$.

Assume that X is a smooth holomorphic variety. Let $\Pi^{\otimes(\bullet+2)} := \ker(\nabla : \mathcal{P}^{\otimes(\bullet+2)} \to |Ome^*_{X^{\Delta[1]}}(\mathcal{P}^{\otimes(\bullet+2)}))$. The seaf $\Pi^{\otimes(\bullet+2)}$ is a locally constant sheaf on $X^{\Delta[1]}$ i.e. each $\Pi^{\otimes(n+2)}$ is locally constant on X^{n+2}. The complex $\Omega^*_{X,\Delta[1]}(\mathcal{P}^{\otimes(\bullet+2)})$ is a resolution of $\Pi^{\otimes(\bullet+2)}$. Hence the spectral sequance of $tR(p^\bullet)_*$ associated with the filtration $\{G^i\}$ converges to $H^*(tRp^\bullet_*\Omega^*\Pi^{\otimes(\bullet+2)})$. The complex $(*q)$ is exact but at a zero term. Hence $ker_1^{0,q} = H^q(tRp^\bullet_*\Omega^*\Pi^{\otimes(\bullet+2)})$.

3.7. Let $\bar{X}$ be a smooth proper scheme of finite type over $\operatorname{Spec} k$ and let D be a divisor with normal crossings in $\bar{X}$. Let $X = \bar{X} \setminus D$. Let $(\bar{\mathcal{P}} \to \bar{X},\ \bar{\nabla} : \bar{\mathcal{P}} \to \Omega^1_{\bar{X}}\langle D\rangle \otimes_{\mathcal{O}_{\bar{X}}} \bar{\mathcal{P}})$ be the canonical extension of $(\mathcal{P} \to X,\ \nabla : \mathcal{P} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{P}))$.

Consider the following morphisms between cosimplicial schemes

$$\begin{array}{ccc} X^{\Delta[1]} & \xrightarrow{j^\bullet} & (\bar{X}^{\Delta[1]} =: \bar{\mathcal{X}} \\ \downarrow p^\bullet & & \downarrow \bar{p}^\bullet \\ (X \times X)^\bullet & \xrightarrow{i^\bullet} & (\bar{X} \times \bar{X})^\bullet \end{array}$$

where the horizontal morphisms are induced by inclusion $X \hookrightarrow \bar{X}$.

Let $D(n) := X \times \bar{X}^n \times X \setminus X \times X^n \times X$ and $Y(n) := X \times \bar{X}^n \times \bar{X} \setminus X \times \bar{X}^n \times X$. Let us set $\bar{\mathcal{X}} := X^\Delta[1]$. The twisted logarithmic De Rham complex on $\bar{\mathcal{X}}$, $\Omega^*_{\bar{\mathcal{X}}}\langle D(\bullet) + Y(\bullet)\rangle(\bar{\mathcal{P}}^{\otimes(\bullet+2)})$ (at degree n we have $\Omega^*_{\bar{X}^{n+2}}\langle D(n) + Y(n)\rangle(\bar{\mathcal{P}}^{\otimes(n+1)})$) and the twisted logarithmic relative De Rham complex on $\bar{\mathcal{X}}$,

$$\Omega^*\langle D(\bullet) + Y(\bullet)\rangle := \Omega^*_{\bar{\mathcal{X}}/(\bar{X}\times\bar{X})^\bullet}\langle D(\bullet) + Y(\bullet)\rangle(\bar{\mathcal{P}}^{\otimes(\bullet+2)})$$

(at degree n we have $\Omega^*_{\bar{X}\times\bar{X}^n\times\bar{X}/y\times\bar{X}}\langle D(n) + Y(n)\rangle \otimes_{\mathcal{O}_{\bar{X}\times\bar{X}^n\times\bar{X}}} (\bar{\mathcal{P}}^{\otimes(n+2)})$) are complexes of sheaves on $\bar{\mathcal{X}}$. Let us set

$$\bar{G}^i_n(\Omega^*_{\bar{X}^{n+2}}\langle D(n) + Y(n)\rangle(\bar{\mathcal{P}}^{\otimes(n+2)})) :=$$

$$\operatorname{image}\big(\Omega^{*-1}_{\bar{X}^{n+2}}\langle D(n)+Y(n)\rangle(\bar{\mathcal{P}}^{\otimes(n+2)})\otimes(p^n)^*\Omega^i_{\bar{X}\times\bar{X}}\langle D\rangle \to \Omega^*_{\bar{X}^{n+2}}\langle D(n)+Y(n)\rangle(\bar{\mathcal{P}}^{\otimes(n+2)})\big).$$

As before we construct a spectral sequence of the functor $tRp^\bullet_*\Omega^*$ associated with the filtration $\{\bar{G}^i\}_i$ and we get $\bar{d}_1^{p,q} : \bar{E}_1^{p,q} \to \bar{E}_1^{p+1,q}$ and $\bar{E}_1^{p,q} = \Omega^p_{\bar{X}\times\bar{X}}\langle D\rangle \otimes_{\mathcal{O}_{\bar{X}\times\bar{X}}} H^q(tRp^\bullet_*\Omega^*\langle D(\bullet) + Y(\bullet)\rangle)$. The differential $\bar{d}_1^{0,q} \to \bar{E}_1^{1,q}$ is an integrable connection on $H^q(tRp^\bullet_*\Omega^*\langle D(\bullet) + Y(\bullet)\rangle)$.

Theorem 3.7.1. *We have*

i) The sheaf $H^q(tRp_*^\bullet\Omega^*\langle D(\bullet)+Y(\bullet)\rangle)$ *is an extention of the sheaf* $H^q(tRp_*^\bullet\Omega^*)$.

ii) The connection

$$\bar{d}_1^{0,q} : H^q(tRp_*^\bullet\Omega^*\langle D(\bullet)+Y(\bullet)\rangle) \to \Omega^1_{barX\times\bar{X}} \otimes_{\mathcal{O}_{barX\times\bar{X}}} H^q(tRp_*^\bullet\Omega^*\langle D(\bullet)+Y(\bullet)\rangle)$$

is an extension of the connection

$$d_1^{0,q} : H^q(tRp_*^\bullet\Omega^*) \to \Omega^1_{X\times X} \otimes_{\mathcal{O}_{X\times X}} H^q(tRp_*^\bullet\Omega^*).$$

Proof. These two statements are obvious because on X, $\Omega^*_{\bar{X}}\langle D\rangle(\mathcal{U}) = \Omega^*_X(\mathcal{U})$ for an open subset $\mathcal{U}$ of X.

§4. Canonical cocycle associated to a variation of mixed Hodge structures.

4.0. Let X be a smooth complex variety and let γ be a smooth path on X from x_0 to z.

Let $M_1(x),\ M_2(x) \in \Omega^1(X)\otimes M_n(C)$. Then $\int_\gamma M_1(x), M_2(x)$ is an iterated integral such that $d(\int_\gamma M_1(x), M_2(x)) = M_1(x)\circ\int_\gamma M_2(x)$.

Let us choose $L(x)$ and $C(x)$ in $\Omega^1(X)\otimes M_n(C)$. Assume that the matrix function $\varphi_\gamma(z)$ satisfies a differential equation

$$d\varphi_\gamma(z) + L(z)\varphi_\gamma(z) = 0,\ \ \varphi_\gamma(x_0) = Id.$$

Then the sum of iterated integrals

$$F_\gamma(z) := \varphi_\gamma(z)(Id + \int_\gamma \varphi_\gamma(x)^{-1}\circ(-C(x))\circ\varphi_\gamma(x) +$$

$$\int_\gamma \varphi_\gamma(x)^{-1}\circ(-C(x))\circ\varphi_\gamma(x), \varphi_\gamma(x)^{-1}\circ(-C(x))\circ\varphi_\gamma(x) +$$

$$\int_\gamma \varphi_\gamma(x)^{-1}\circ(-C(x))\circ\varphi_\gamma(x), \varphi_\gamma(x)^{-1}\circ(-C(x))\circ\varphi_\gamma(x), \varphi_\gamma(x)^{-1}\circ(-C(x))\circ\varphi_\gamma(x) + \dots$$

satisfies a differential equation

$$dF_\gamma(z) + (L(z)+C(z))F_\gamma(z) = 0,\ \ F_\gamma(x_0) = Id.$$

Let us set

$$f_0(z) := \varphi_\gamma(z), \quad f_1(z) := \varphi_\gamma(z) \int_\gamma \varphi_\gamma(x)^{-1} \circ (-C(x)) \circ \varphi_\gamma(x),$$

$$\dots$$

$$f_n(z) := \varphi_\gamma(z) \int_\gamma \varphi_\gamma(x)^{-1} \circ (-C(x)) \circ \varphi_\gamma(x), \varphi_\gamma(x)^{-1} \circ (-C(x)) \circ \varphi_\gamma(x), \dots.$$

Let $g_k(z) = \sum_{i=0}^k f_i(z)$. The functions $g_k(z)$ satisfy the following system of differential equations

$$dg_0(z) + L(z)g_0(z) = 0, \quad g_0(x_0) = Id;$$

$$dg_1(z) + L(z)g_1(z) + C(z)g_0(z) = 0, \quad g_1(x_0) = Id;$$

$$\dots$$

$$dg_n(z) + L(z)g_n(z) + C(z)g_{n-1}(z) = 0, \quad g_n(x_0) = Id.$$

Assume that $f_i(z) = 0$ for $i \geq n+1$. Then $g_n = g_{n+1}$ and

$$dg_n(z) + (L(z) + C(z))g_n(z) = 0.$$

Hence the columns of $g_n(z)$ are flat sections of the connection

$$\nabla : \mathcal{O}_X^n \to \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{O}_X^n$$

given by $\nabla(s) = ds + (L + C)s$.

Let γ_1 be a path from z to z'. Then we have

$$F_{\gamma_1 \circ \gamma}(z') = F_\gamma(z') \cdot F_\gamma(z).$$

For an iterated integral of length n we get

$$\varphi_{\gamma_1 \cdot \gamma}(z') \int_{\gamma_1 \cdot \gamma} (\varphi_{\gamma_1 \cdot \gamma}(x))^{-1} \circ (-C_n(x)) \circ \varphi_{\gamma_1 \cdot \gamma}(x), \dots, (\varphi_{\gamma_1 \cdot \gamma}(x))^{-1} \circ (-C_1(x)) \circ \varphi_{\gamma_1 \cdot \gamma}(x) =$$

$$\varphi_{\gamma_1}(z')\circ\varphi_\gamma(z)\big(\sum_{i=0}^{n}\big(\int_{\gamma_1}(\varphi_\gamma(z))^{-1}\circ(\varphi_{\gamma_1}(x))^{-1}\circ(-C_n(x))\circ\varphi_{\gamma_1}(x)\circ\varphi_\gamma(z),\dots,$$

$$(\varphi_\gamma(z))^{-1}\circ(\varphi_{\gamma_1}(x))^{-1}(-C_{i+1}(x))\circ\varphi_{\gamma_1}(x)\circ\varphi_\gamma(z)\big)\circ$$

$$\big(\int_\gamma(\varphi_\gamma(x))^{-1}\circ(-C_i(x))\circ\varphi_\gamma(x),\dots,(\varphi_\gamma(x))^{-1}\circ(-C_1(x))\circ\varphi_\gamma(x)\big)\big)$$

where $C_i(x)=C(x)$.

We shall use these observations to associate to a variation of mixed Hodge structures over X a mixed Hodge representation of the Hopf algebra $\mathcal{H}$. The definition of comultiplication $\mathcal{M}:\mathcal{H}\to\mathcal{H}\otimes\mathcal{H}$ was motivated by the last equality.

4.1. We suppose for simplicity that the principal G-bundle $\pi:P\to X$ is trivial.

Let $\pi:X\times G\to X$ be a principal G-bundle equipped with the integrable connection given by a one-form $\omega\in\Omega^1(X)\otimes\mathfrak{g}$, where $\mathfrak{g}=\mathrm{Lie}G$.

Let $\rho:G\to\mathrm{Aut}_{\mathrm{algebra}}(\mathcal{O}[G])$ be given by $g(f):={}^gf$, where $({}^gf)(x):=f(g^{-1}\cdot x)$. Let $\tilde\rho:\mathfrak{g}\to\mathrm{Der}_{\mathrm{algebra}}(\mathcal{O}[G])$ be the induced map of Lie algebras.

Lemma 4.1.1. *Let $\mu:\mathcal{O}[G]\to\mathcal{O}[G]\otimes\mathcal{O}[G]$ be induced by the multiplication $G\times G\to G$, $(g_2,g_1)\to g_2\cdot g_1$. Let us set $\mu(f)(g_2,g_1)=\sum_i\mu(f)^i_2(g_2)\otimes\mu(f)^i_1(g_1)$. Let $X\in\mathfrak{g}$. Then we have*

$$\mu(\tilde\rho(X)(f))=\sum_i\tilde\rho(X)(\mu(f)^i_2)\otimes\mu(f)^i_1.$$

Proof. It follows from the equality $\mu({}^gf)=\sum_i{}^g(\mu(f)^i_2)\otimes\mu(f)^i_1$.

Let $\varphi:G\to\mathrm{Aut}(V)$ be a representation of G and let $\tilde\varphi:\mathfrak{g}\to\mathrm{End}(V)$ be the induced representation of the Lie algebra $\mathfrak{g}$. Let us fix a base $e_1,\dots,e_n$ of V. Let $\varphi(g)=(a_{ij}(g))_{i,j}$ in this base. Let us set:

$$\tau_\varphi:=(a_{ij})_{i,j}\in\mathcal{O}[G]\otimes\mathrm{End}(V).$$

The group G acts on $\mathcal{O}[G]$, hence it acts also on $\mathcal{O}[G]\otimes\mathrm{End}(V)$. We have

$$g:(a_{i,j})_{i,j}\to(a_{i,j}(g^{-1}))_{i,j}\cdot(a_{i,j})_{i,j}=(\sum_k a_{i,k}(g^{-1})a_{k,j})_{i,j}.$$

The subspace $\tau_\varphi^{-1}\mathrm{End}(V)\tau_\varphi\subset\mathcal{O}[G]\otimes\mathrm{End}(V)$ is G-invariant.

Lemma 4.1.2. *Let $X \in \mathfrak{g}$ and $C \in \mathrm{End}(V)$. We have*

$$\tilde{\rho}(X)(\tau_\varphi^{-1} C \tau_\varphi) = \tau_\varphi^{-1}[\tilde{\varphi}(X), C]\tau_\varphi,$$

$$\tilde{\rho}(X)(\tau_\varphi^{-1} C) = \tau_\varphi^{-1}\tilde{\varphi}(X)C, \;\; \tilde{\rho}(X)(C\tau_\varphi) = -C\tilde{\varphi}(X)\tau_\varphi.$$

Proof. Observe that the subspaces $\tau_\varphi^{-1}\mathrm{End}(V)$ and $\mathrm{End}(V)\tau_\varphi$ of $\mathcal{O}[G] \otimes \mathrm{End}(V)$ are also G-invariant. The group G acts on $\tau_\varphi^{-1}\mathrm{End}(V)$. The element $g \in G$ maps $(a_{i,j})_{i,j}^{-1} \circ C$ into $(a_{i,j})_{i,j}^{-1} \circ (a_{k,l}(g))_{k,l} \circ C$. Let $X \in \mathfrak{g}$. Then $\tilde{\rho}(X)((a_{i,j})_{i,j}^{-1} \circ C) = \lim_{t\to 0} \frac{1}{t}((a_{i,j})_{i,j}^{-1} \circ (a_{i,j}(\exp tX))_{i,j} \circ C - (a_{i,j})_{i,j}^{-1} \circ C) = (a_{i,j})_{i,j}^{-1} \circ \tilde{\varphi}(X) \circ C$.

4.2. The associated vector bundle

$$(X \times G) \times_\rho \mathcal{O}[G] \to X$$

equipped with the connection given by a one-form $\tilde{\rho}(\omega) \in \Omega^1(X) \otimes \mathrm{Der}_{algebra}(\mathcal{O}[G])$ is the pair $(\mathcal{P} = \pi_* \mathcal{O}_P \to X, \nabla)$ from section 3.1. Let $f \in \mathcal{O}(X)$ and $\chi \in \mathcal{O}[G]$. Then

4.2.0. $$\nabla(f \otimes \chi) = df \otimes \chi + f\tilde{\rho}(\omega)(\chi).$$

The connection ∇ defines trivially a connection also denoted by ∇, on the vector bundle

$$((X \times G) \times_\rho \mathcal{O}[G]) \otimes \mathrm{End}(V) \to X.$$

Lemma 4.2.1. *Let $C(z) \in \mathcal{O}(X) \otimes \mathrm{End}(V)$. We have*

$$\nabla(\tau_\varphi^{-1} C(z)\tau_\varphi) = \tau_\varphi^{-1}(dC(z) + [\tilde{\varphi}(\omega), C(z)])\tau_\varphi.$$

Proof. The lemma follows from the formula 4.2.0 and Lemma 4.1.2.

4.3. Let $\mathcal{V}$ be a vector bundle on X equipped with the integrable connection $\nabla_{\mathcal{V}} : \mathcal{V} \to \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{V}$. We shall assume that $\mathcal{V}$ is trivial and that it is equipped with the increasing filtration $\{W_i\}_i$ compatible with the connection, by trivial sub-bundles. We assume the associate graded bundle with the induced connection is a bundle

$$(X \times G) \times_\varphi V \to X,$$

for a certain representation $\varphi : G \to \mathrm{Aut}V$, equipped with a connection given by a one-form $L(z) := \tilde{\varphi}(\omega) \in \Omega^1(X) \otimes \mathrm{End}(V)$. Hence the connection $\nabla_{\mathcal{V}}$ is given by a one-form

$$L(z) + C(z) \in \Omega^1(X) \otimes \mathrm{End}(V).$$

Observe that $C(z)^n = 0$ if the filtration of has a length n. We have

$$dL(z) + L(z) \wedge L(z) = 0.$$

The integrability of the connection $\nabla_{\mathcal{V}}$ implies

4.3.1. $$dC(z) + C(z) \wedge L(z) + L(z) \wedge C(z) + C(z) \wedge C(z) = 0.$$

4.4. Let X be a smooth holomorphic variety and let $\Omega^*(X)$ be the De Rham complex of smooth complex values differential forms on X or let X be an affine algebraic variety over k and let $\Omega^*(X)$ be the algebraic De Rham complex on X.

We recall from 2.5 that $T_{x,x}$ is the total complex of the double complex

$$\bigoplus_{n=0}^{\infty} \mathcal{O}[G] \otimes \Omega^*(\mathcal{P})(X)^{\otimes n} \otimes \mathcal{O}[G] = \bigoplus_{n=0}^{\infty} \mathcal{O}[G] \otimes (\Omega^*(X) \otimes \mathcal{O}[G])^{\otimes n} \otimes \mathcal{O}[G].$$

Definition 4.4.1. Let us set

$$\tau^{(n)} := \tau_\varphi(g_n)^{-1} \circ (-C(z_n)) \circ \tau_\varphi(g_n) \otimes \ldots \otimes \tau_\varphi(g_1)^{-1} \circ (-C(z_1)) \circ \tau_\varphi(g_1) \otimes \tau_\varphi(g_0)^{-1}$$
$$\in (\mathcal{O}[G] \otimes (\Omega^*(X) \otimes \mathcal{O}[G])^{\otimes n} \otimes \mathcal{O}[G]) \otimes \mathrm{End}(V)$$

and

$$\tau_\nabla = \tau_{\nabla_{\mathcal{V}}} := \tau^{(0)} + \tau^{(1)} + \ldots + \tau^{(n)} + \ldots.$$

Observe that only a finite number of $\tau^{(i)}$ are non-zero, hence $\tau_{\nabla_{\mathcal{V}}} \in T^0_{x,x} \otimes \mathrm{End}(V)$.

Lemma 4.4.2. *The element $\tau_{\nabla_{\mathcal{V}}}$ is a cocycle.*

Proof. We have $(-1)^n \delta(\tau^{(n)}) = (-1)^n \sum_{i=1}^{n-1} (-1)^i \ldots \otimes \tau_\varphi(g_i)^{-1} \circ (C(z_i) \wedge C(z_i)) \circ \tau_\varphi(g_i) \otimes$

$$\tau_\varphi(g_{i-1})^{-1} \circ (-C(z_{i-1})) \circ \tau_\varphi(g_{i-1}) \otimes \ldots \otimes \tau_\varphi(g_1)^{-1} \circ (-C(z_1)) \circ \tau_\varphi(g_1) \otimes \tau_\varphi(g_0)^{-1}$$

and

$$\partial(\tau^{(n-1)}) = \sum_{i=1}^{n-1} (-1)^{n-i-1} \ldots \otimes \tau_\varphi(g_i)^{-1} \circ (-dC(z_i) - L(z_i)C(z_i) - C(z_i)L(z_i)) \circ \tau_\varphi(g_i) \otimes \ldots.$$

Hence it follows from 4.3.1 that $\tau_{\nabla_{\mathcal{V}}}$ is a cocycle.

Lemma 4.4.3. *We have*

$$(id_{T^0_{x,x}} \otimes \tau_{\nabla_{\mathcal{V}}}) \circ \tau_{\nabla_{\mathcal{V}}} = (\mathcal{M} \otimes id_V) \circ \tau_{\nabla_{\mathcal{V}}}.$$

Proof. We recall that for a trivial bundle $\pi : X \times G \to X$ the comultiplication $\mathcal{M} : T^0_{x,x} \to T^0_{x,x} \otimes T^0_{x,x}$ is given by

$$\mathcal{M}(f_{n+1}(g_{n+1}) \otimes \omega_n f_n(g_n) \otimes \ldots \otimes \omega_1 f_1(g_1) \otimes f_0(g_0)) =$$

$$\sum_{i=0}^{n} (f_{n+1}(g_{n+1}) \otimes \omega_n f_n(g_n) \otimes \ldots \otimes \omega_{i+1} f_{i+1}(g_{i+1}) \otimes 1) \otimes (1 \otimes \omega_i f_i(g'_i g_i) \otimes \ldots \otimes \omega_1 f_1(g'_1 g_i) \otimes f_0(g'_0 g_i))$$

Hence we have

$$(\mathcal{M} \otimes id_V) \circ \tau^{(n)}(g_n, \ldots, g_1, g_0) = \sum_{i=0}^{n} \big(\tau_\varphi(g_n)\big)^{-1} \circ (-C(z_n)) \circ \tau_\varphi(g_n) \otimes \ldots$$

$$\otimes (\tau_\varphi(g_{i+1}))^{-1} \circ (-C(z_{i+1})) \circ \tau_\varphi(g_{i+1}) \otimes id_V\big) \otimes \big(id_V \otimes \tau_\varphi(g'_i g_i))^{-1} \circ (-C(z_i)) \circ \tau_\varphi(g'_i g_i) \otimes \ldots$$

$$\otimes (\tau_\varphi(g'_1 g_i))^{-1} \circ (-C(z_1)) \circ \tau_\varphi(g'_1 g_i) \otimes \tau_\varphi(g'_0 g_i)^{-1}\big) =$$

$$\sum_{i=0}^{n} \tau^{(n-i)}(g_n, \ldots, g_{i+1}, g_i) \otimes \tau^{(i)}(g'_i, \ldots, g'_1, g'_0).$$

This implies the lemma.

Hence the class of $\tau_{\nabla_{\mathcal{V}}}$, which for simplicity, we also denote by $\tau_{\nabla_{\mathcal{V}}}$ is a representation of $\mathcal{H}$. We shall show that $\tau_{\nabla_{\mathcal{V}}}$ is a mixed Hodge representation. First however we drop the assumption that the principal G-bundle $\pi : P \to X$ and the vector bundle $\mathcal{V}$ are trivial. We shall work in Cech cohomology.

4.5. Let $\pi : P \to X$ be a principal G-bundle equipped with the integrable connection given by a one-form $\omega \in \Omega^1(P) \otimes \mathfrak{g}$. Let $\mathcal{U} = \{U_i\}_{i \in J}$ be an open covening of X and let $\{\psi_i : \pi^{-1}(U_i) \to U_i \times G\}_{i \in J}$ be a family of G-isomorphisms. Let

$$\{\psi_{j,i} : U_j \cap U_i \to G\}$$

be the family of transition functions. They satisfy the cocycle condition

4.5.1. $$\psi_{ki}(x) = \psi_{kj}(x) \cdot \psi_{ji}(x)$$

on $U_k \cap U_j \cap U_i$. For each i, let $\sigma_i : U_i \to P$ be a section on U_i defined by $\sigma_i(x) = \psi_i^{-1}(x, e)$. Let θ be the left invariant $\mathfrak{g}$-valued canonical 1-form on G.

For each i, we define a $\mathfrak{g}$-valued one-form ω_i on U_i by $\omega_i := (\sigma_i)^*\omega$. The forms $\{\omega_i\}_{i\in J}$ satisfy

4.5.2. $$\omega_j = ad(\psi_{ij}^{-1})\omega_i + \psi_{ij}^*\theta$$

on $U_i \cap U_j$ and they determine the connection form ω.

Remark 4.5.3. Observe that the description of a principal G-bundle equipped with a connection as a family of transition functions satisfying cocycle condition and a family of one-forms satisfying 4.5.2 is valid in étale topology or flat topology.

4.6. Let $\rho : G \to \mathrm{Aut}_{algebra}(\mathcal{O}[G])$ be given by $g(f) := {}^g f$, where $({}^g f)(x) := f(g^{-1}x)$. Then $\{\Psi_{ji} := \rho \circ \psi_{ji}\}_{ji}$ are transition functions of the associated vector bundle

$$\mathcal{P} := (P \times_G \mathcal{O}[G] \to X).$$

Let $\tilde{\rho} : \mathfrak{g} \to \mathrm{Der}_{algebra}(\mathcal{O}[G])$ be the induced map of Lie algebras. Let us set

$$\Lambda_i := \tilde{\rho}(\omega_i) \in \Omega^1(U_i) \otimes \mathrm{Der}_{algebra}(\mathcal{O}[G]).$$

Then it follows from 4.5.2 that

4.6.1. $$\Psi_{ji}\Lambda_i = \Lambda_j\Psi_{ji} + d\Psi_{ji}.$$

Hence the family of one-forms $\{\Lambda_i\}_{i\in J}$ defines an integrable multiplicative connection on $\mathcal{P}$. This is the connection ∇ from section 3.1.

4.7. Let $\varphi : G \to \mathrm{Aut}V$ be a representation of G. The functions $\{A_{ji} := \varphi \circ \psi_{ji}\}_{ji}$ satisfy the cocycle condition. They are transition functions of the associated vector bundle $\mathcal{L} := (P \times_\varphi V \to X)$. Let $\tilde{\varphi} : \mathfrak{g} \to \mathrm{End}(V)$ be the induced map of Lie algebras. Let us set

$$\lambda_i := \tilde{\varphi}(\omega_i) \in \Omega^1(U_i) \otimes \mathrm{End}(V).$$

It follows from 4.5.2 that

4.7.1. $$A_{ji}\lambda_i = \lambda_j A_{ji} + dA_{ji}.$$

The one-forms $\{\lambda_i\}_{i\in J}$ determine an integrable connection $\nabla_{\mathcal{L}}$ on $\mathcal{L}$.

4.7.2. We assume that $\mathcal{L}$ is a graded vector bundle and the connection $\nabla_{\mathcal{L}}$ is compatible with the gradation i.e.

$$(\mathcal{L}, \nabla_{\mathcal{L}}) = (\oplus_\alpha \mathcal{L}_\alpha, \oplus_\alpha \nabla_{\mathcal{L}_\alpha}).$$

The transition functions $\{A_{ji}\}$ and the one-forms $\{\omega_i\}$ are compatible with the gradation.

4.8. Let $\chi_{ji} : U_j \cap U_i \to \mathrm{Aut}(V)$ $((j,i) \in J \times J)$ be a family of functions satisfying a cocycle condition. Let $M_i \in \Omega^1(U_i) \otimes \mathrm{End}(V)$ $(i \in J)$ be a family of one-forms such that

4.8.1. $$\chi_{ji} \circ M_i = M_j \circ \chi_{ji} + d\chi_{ji}.$$

The cocycle $\{\chi_{ji}\}$ and the family of one-forms $\{M_i\}$ define a vector bundle $\mathcal{V}$ and a connection $\nabla_{\mathcal{V}}$. We assume that the connection $\nabla_{\mathcal{V}}$ is integrable.

We assume that the vector bundle $\mathcal{V}$ is equipped with the filtration $\{W_\alpha \mathcal{V}\}_\alpha$ compatible with the transition functions $\{\chi_{ji}\}$ and the connection one-forms $\{M_i\}$. We assume further that the associated graded vector bundle with the induced connection is $(\oplus \mathcal{L}_i, \oplus \nabla_{\mathcal{L}_i})$. In terms of transition functions and connection matrices it means:

$$\chi_{ji} = A_{ji} + B_{ji}, \quad M_i = \lambda_i + C_i$$

and B_{ji} and C_i vanish when passing to the associated graded vector bundle.

4.9. We recall that a Cech complex $C^*(\mathcal{U}; \Omega^*(\mathcal{P}))$ associated with a covering $\mathcal{U} = \{U_i\}_{i\in J}$ is defined in the following way:

$$C^m(\mathcal{U}; \Omega^n(\mathcal{P})) := \{f(i_0, \ldots, i_m) \in \Omega^n(\mathcal{P})(U_{i_0} \cap \ldots U_{i_m})\}_{(i_0,\ldots,i_m)\in J^m};$$

$$\{(Df)(i_0, \ldots, i_{m+1})\} := \{\sum_{k=0}^{m+1} (-1)^k f(i_0, \ldots, \hat{i_k}, \ldots, i_{m+1})\};$$

∂ is induced by the differential of twisted De Rham complex;

$$C^*(\mathcal{U}; \Omega^*(\mathcal{P})) := Tot(\bigoplus_{n,m=0}^{\infty} C^m(\mathcal{U}; \Omega^n(\mathcal{P})))$$

with a differential d equal $D + (-1)^m \partial$ on $C^m(\mathcal{U}; \Omega^n(\mathcal{P}))$.

The trivilization $\{\psi_i : \pi^{-1}(U_i) \to U_i \times G\}_{i\in J}$ of the principal G-bundle $\pi : P \to X$ induces an isomorphism

$$\mathcal{O}_X(U_i) \otimes \mathcal{O}[G] \approx \Omega^0(\mathcal{P})(U_i).$$

We define elements $\tau_{\varphi_i} \in \Omega^0(\mathcal{P})(U_i) \otimes \mathrm{End}(V)$ by setting

$$\tau_{\varphi_i} := 1 \otimes \tau_\varphi \in \mathcal{O}_X(U_i) \otimes \mathcal{O}[G] \otimes \mathrm{End}(V).$$

Let us define an element $\sigma(\nabla_{\mathcal{V}})$ in

$$C^1(\mathcal{U}; \Omega^0(\mathcal{P})) \otimes \mathrm{End}(V) \oplus C^0(\mathcal{U}; \Omega^1(\mathcal{P})) \otimes \mathrm{End}(V)$$

by the following formula

$$\sigma(\nabla_{\mathcal{V}}) = \sigma := b + c := \{(j,i) \to \tau_{\varphi_j}^{-1} \circ (-B_{ji}) \circ \tau_{\varphi_i} \in \Omega^0(\mathcal{P})(U_j \cap U_i) \otimes \mathrm{End}(V)\}+$$

$$\{(i) \to \tau_{\varphi_i}^{-1} \circ (-C_i) \circ \tau_{\varphi_i} \in \Omega^1(\mathcal{P})(U_i) \otimes \mathrm{End}(V)\}.$$

Lemma 4.9.1. *We have*

$$\partial c - c \wedge c = 0 \text{ in } C^0(\mathcal{U}; \Omega^2(\mathcal{P})) \otimes \mathrm{End}(V);$$

$$Db - b \cup b = 0 \text{ in } C^2(\mathcal{U}; \Omega^0(\mathcal{P})) \otimes \mathrm{End}(V);$$

$$\partial b + c \cup b + b \cup c - Dc = 0 \text{ in } C^2(\mathcal{U}; \Omega^0(\mathcal{P})) \otimes \mathrm{End}(V).$$

Proof. The first equality follows from the fact that the connections $\nabla_{\mathcal{L}}$ and $\nabla_{\mathcal{V}}$ are integrable. The fact that $\{\chi_{ji}\}$ is a cocycle implies the second equality. The third one follows from the formula 4.8.1.

Lemma 4.9.2. *We have*

$$d\sigma(\nabla_{\mathcal{V}}) = \sigma(\nabla_{\mathcal{V}}) \cup \sigma(\nabla_{\mathcal{V}}) \text{ in } C^*(\mathcal{U}; \Omega^*(\mathcal{P})) \otimes \mathrm{End}(V).$$

Proof. The lemma follows from Lemma 4.9.1.

Let $\chi'_{ji} : U_j \cap U_i \to \mathrm{Aut}(V')$ be another family of functions satisfying a cocycle condition and let $M'_i \in \Omega^1(U_i) \otimes \mathrm{End}(V')$ be a family of 1-forms satisfying 4.8.1. Assume that the vector bundle $\mathcal{V}'$ determined by the cocyle χ'_{ji} and the connection $\nabla_{\mathcal{V}'}$ defined by the family $\{M'_i\}$ satisfy the condition 4.8. In terms of transition functions and connection matrices it means

$$\chi'_{ji} = A'_{ji} + B'_{ji} \;\; M'_i = \lambda'_i + C'_i$$

and B'_{ij} and C'_i vanish when passing to the associated graded vector bundle. The associated graded bundle $\mathcal{L}'$ is $P \times_{\varphi'} V' \to X$, where $\varphi' : G \to \mathrm{Aut}(V')$ is a representation of G, $A'_{ji} = \varphi' \circ \psi_{ij}$ and $\lambda'_i = \tilde{\varphi}'(\omega_i)$.

We assume that the triples $(\mathcal{V}, \{W_\alpha \mathcal{V}\}_\alpha, \nabla_{\mathcal{V}})$ and $(\mathcal{V}', \{W_\alpha \mathcal{V}'\}_\alpha, \nabla_{\mathcal{V}'})$ are isomorphic. Hence we have an isomorphism

$$(\mathcal{L}, \nabla_{\mathcal{L}}) \approx (\mathcal{L}', \nabla_{\mathcal{L}'}).$$

Let $f : V \to V'$ be an isomorphism of fibers over x. The fact that the image of the monodromy representation $\Theta_x : \pi_1(X, x) \to G$ is Zariski dense in G implies that

$$\varphi'(g) = f \circ \varphi(g) \circ f^{-1}$$

for all $g \in G$.

Locally the isomorphism $(\mathcal{V}, \{W_\alpha \mathcal{V}\}_\alpha, \nabla_{\mathcal{V}}) \approx (\mathcal{V}', \{W_\alpha \mathcal{V}'\}_\alpha, \nabla_{\mathcal{V}'})$ means that there are $h_i \in \mathcal{O}_X(\mathcal{U}_i) \otimes \mathrm{Hom}(V', V)$ such that

$$h_j \chi'_{ji} = \chi_{ji} h_i \;\text{ and }\; h_i M'_i = M_i h_i + dh_i.$$

Let us set

$$\chi''_{ji} = f^{-1} \chi'_{ji} f$$

and in general

$$(\ )'' = f^{-1}(\ )'f.$$

Observe that $A''_{ji} = A_{ji}$ and $\lambda''_i = \lambda_i$. The cocycle $\{\chi''_{ji}\}$ and a family $\{M''_i\}$ define a vector bundle $\mathcal{V}''$ and an integrable connection $\nabla_{\mathcal{V}''}$.

Let us set

$$k_i := h_i \circ f \in \mathcal{O}_X(U_i) \otimes \mathrm{End}(V).$$

We have

$$k_j \chi''_{ji} = \chi_{ji} k_i \text{ and } k_i M''_i = M_i k_i + dk_i.$$

Hence the triples $(\mathcal{V}, \{W_\alpha \mathcal{V}\}, \nabla_{\mathcal{V}})$ and $(\mathcal{V}'', \{W_\alpha \mathcal{V}''\}, \nabla_{\mathcal{V}''})$ are isomorphic. Let

$$\eta \in C^0(\mathcal{U}; \Omega^0_X(\mathcal{P})) \otimes \mathrm{End}(V)$$

be given by:

$$\eta = \{(i) \to \tau^{-1}_{\varphi_i} \circ (k_i - Id) \circ \tau_{\varphi_i} \in \Omega^0_X(\mathcal{P})(U_i) \otimes \mathrm{End}(V)\}.$$

Lemma 4.9.3. *We have*

$$\sigma - \sigma'' = d\eta - \sigma \cup \eta + \eta \cup \sigma'', \text{ where } \sigma = \sigma_{\nabla_{\mathcal{V}}} \text{ and } \sigma'' = \sigma_{\nabla_{\mathcal{V}''}}.$$

Lemma 4.9.4. *Let us set $\sigma\prime = \sigma_{\nabla_{\mathcal{V}'}}$. We have $\sigma' = f \circ \sigma'' \circ f^{-1}$.*

4.10. Now we shall compute Cech cohomology of $\mathcal{Y}_{yx} := (X; x, y)^{(\Delta[1];0,1)}$. Let

$$\partial : \mathcal{O}[G] \otimes C^*(\mathcal{U}; \Omega^*(\mathcal{P}))^{\otimes n} \otimes \mathcal{O}[G] \to \mathcal{O}[G] \otimes C^*(\mathcal{U}; \Omega^*(\mathcal{P}))^{\otimes n} \otimes \mathcal{O}[G]$$

be the differential of a tensor product of complexes $C^*(\mathcal{U}; \Omega^*(\mathcal{P}))$.

The two augmentations ε_x and η_x are non-zero only on $C^0(\mathcal{U}; \Omega^0(\mathcal{P})) = \prod_{i \in J} \mathcal{P}(U_i)$. We choose i_0 such that $x \in U_{i_0}$. The augmentation are compositions of the projection of $C^0(\mathcal{U}; \Omega^0(\mathcal{P}))$ onto $\mathcal{P}(U_{i_0})$ with ε_x and η_x defined in

Differentials

$$\delta : \mathcal{O}[G] \otimes C^*(\mathcal{U}; \Omega^*(\mathcal{P}))^{\otimes n} \otimes \mathcal{O}[G] \to \mathcal{O}[G] \otimes C^*(\mathcal{U}; \Omega^*(\mathcal{P}))^{\otimes (n-1)} \otimes \mathcal{O}[G]$$

are defined using structure maps $\delta^i : y \times X^{n-1} \times x \to y \times X^n \times x \quad i = 0, 1, \ldots, n$ and augmentations. Let us set:

$$C^*(\mathcal{U}; \mathcal{O}[G] \otimes \Omega^*_{\mathcal{Y}_{yx}}(\mathcal{P}^\bullet) \otimes \mathcal{O}[G]) := Tot(\bigoplus_{n=0}^{\infty} \mathcal{O}[G] \otimes C^*(\mathcal{U}; \Omega^*(\mathcal{P})^{\otimes n} \otimes \mathcal{O}[G])$$

with a differential d obtain from ∂ and δ with the same sign convention as in 4.9.

Definition 4.10.1. Let us set:

$$\tau^{(n)}_{\nabla_{\mathcal{V}}} := \sigma(\nabla_{\mathcal{V}})(g_n, z_n) \otimes \ldots \sigma(\nabla_{\mathcal{V}})(g_1, z_1) \otimes \tau_\varphi(g_0)^{-1} \in C^*(\mathcal{U}; \mathcal{O}[G] \otimes \Omega^*(\mathcal{P})^{\otimes n} \otimes \mathcal{O}[G]) \otimes \mathrm{End}(V)$$

and

$$\tau_{\nabla_{\mathcal{V}}} := \tau^{(0)}_{\nabla_{\mathcal{V}}} + \tau^{(1)}_{\nabla_{\mathcal{V}}} + \ldots + \tau^{(n)}_{\nabla_{\mathcal{V}}} + \ldots \in Tot(\bigoplus_{n=0}^{\infty} \mathcal{O}[G] \otimes C^*(\mathcal{U}; \Omega^*(\mathcal{P})^{\otimes n}) \otimes \mathcal{O}[G]) \otimes \mathrm{End}(V).$$

Lemma 4.10.2. *The element $\tau_{\nabla_{\mathcal{V}}}$ is a cocycle.*

Proof. The lemma follows from Lemma 4.9.2.

Let us set

$$\zeta(j, i) := \sigma(g_j) \otimes \ldots \otimes \sigma(g_1) \otimes \eta(g) \otimes \sigma''(h_i) \otimes \ldots \otimes \sigma''(h_1) \otimes \tau_\varphi(g_0).$$

Without lost of generality we can assume that the family $\{h_i\}$ is such that $h_{i_0}(x) = Id$.

Lemma 4.10.3. *We have*

$$d(\sum_{i,j=0}^{\infty} (-1)^j \zeta(j, i)) = \tau_{\nabla_{\mathcal{V}}} - \tau_{\nabla_{\mathcal{V}''}}.$$

Proof. It follows from Lemma 4.9.3.

Lemma 4.10.3. *We have*

$$\tau_{\nabla_{\mathcal{V}'}} = f \circ \tau_{\nabla_{\mathcal{V}}} \circ f^{-1}.$$

Proof. It follows from Lemma 4.9.4.

4.11. Mixed Hodge representations.

Definition 4.11.1. Let V be a vector space carring a mixed Hodge structure. Let H be a Hopf algebra equipped with a mixed Hodge structure such that the structures maps are

morphism of mixed Hodge structures. We say that a representation $\rho : V \to H \otimes V$ of the Hopf algebra H is a mixed Hodge representation if ρ is a morphism of mixed Hodge structures.

We shall show that the class $\tau_{\nabla_{\mathcal{V}}} : V \to \mathcal{H} \otimes V$ constructed in sections 4.4 (for trivial bundles) and 4.10 (in general case) is a mixed Hodge representation. First we consider the case when $\pi : P \to X$ and the vector bundle $\mathcal{V}$ are trivial (see sections 4.1-4.4).

4.11.2. We assume that $\mathcal{O}[G]$ and $\mathcal{P}$ satisfy 3.2.2. We assume that the vector bundle $\mathcal{V}$ carry a variation of mixed Hodge structures such that the triple

$$(\mathcal{V}, \{W_i\}_i, \nabla_{\mathcal{V}} : \mathcal{V} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{V})$$

is such as in 4.3. We assume that the representation $\varphi : GrV = V \to \mathcal{O}[G] \otimes (GrV) = \mathcal{O}[G] \otimes V$ is a Hodge representation. (V is a fiber of $\mathcal{V}$ at x and it is canonically equal $GrV = \oplus_i Gr^i_W V$). This implies that the variation of Hodge structures $Gr\mathcal{V} = \oplus_i Gr^i_W \mathcal{V}$ is in VHS_Θ ($\Theta : \pi_1(X, x) \to G$ is the monodromy homomorphism of the principal G-bundle $\pi : X \times G \to X$).

Proposition 4.11.3. *Let X be a smooth projective complex variety. Let $\mathcal{V}$ be a variation of mixed Hodge structures over X satysfying 4.11.2. Then the representation $\tau_{\nabla_{\mathcal{V}}} : V \to \mathcal{H} \otimes V$ of $\mathcal{H}$ is a mixed Hodge representation.*

Proof. The matrix $C(z) \in \Omega^1(X) \otimes W_{-1}\mathrm{End}(V)$ because $C(z)$ is nilpotent with respect to the weight filtration of $\mathcal{V}$. The representation $\tau_\varphi \in \mathcal{O}[G] \otimes \mathrm{End}(V)$ is a Hodge representation compatible with the filtration $\{W_i\}_i$ of V, hence $\tau_\varphi \in \mathcal{O}[G] \otimes W_0(\mathrm{End}(V))$. Therefore $\tau^{(n)} \in W_n(T^0) \otimes W_{-n}(\mathrm{End}(V))$. This implies $\tau_{\nabla_{\mathcal{V}}} \in W_0(\mathcal{H} \otimes \mathrm{End}(V))$. Observe that $\tau_\varphi \in F^0(\mathcal{O}[G] \otimes \mathrm{End}(V))$. The matrix $C(z) \in \Omega^1(X) \otimes F^{-1}(\mathrm{End}(V))$ because the connection $\nabla_{\mathcal{V}}$ satisfies $\nabla_{\mathcal{V}}(F^p\mathcal{V}) \subset \Omega^1_X \otimes_{\mathcal{O}_X} (F^{p-1}\mathcal{V})$. Hence $\tau_{\nabla_{\mathcal{V}}} \in F^0(\mathcal{H} \otimes \mathrm{End}(V))$.

Now we consider the general case. The condition: "the triple $(\mathcal{V}, \{W_i\}_i, \nabla_{\mathcal{V}} : \mathcal{V} \to \Omega^1_X \otimes_{\mathcal{O}_x} \mathcal{V})$ is such as in 4.3" we replace by "the triple $(\mathcal{V}, \{W_i\}_i, \nabla_{\mathcal{V}} : \mathcal{V} \to \Omega^1_X \otimes_{\mathcal{O}_x} \mathcal{V})$ is such as in 4.8."

Proposition 4.11.4. *Let X be a smooth projective complex variety. Let $\mathcal{V}$ be a variation of mixed Hodge structures over X satisfying 4.11.2. Then the representation $\tau_{\nabla_{\mathcal{V}}} : V \to \mathcal{H} \otimes V$ of $\mathcal{H}$ is a mixed Hodge representation.*

Proof. Observe that $C_j \in |Ome^1(U_j) \otimes W_{-1}\mathrm{End}(V)$ and$B_{ji} \in \Omega^0(U_j \cap U_i) \otimes W_{-1}\mathrm{End}(V)$ becauseC_j and B_{ji} are nilpotent with respect to the weight filtration of $\mathcal{V}$.

4.12. The element $\tau_{\nabla_{\mathcal{V}}} \in \mathcal{H} \otimes \mathrm{End}(V)$. We recall that $\mathcal{H} \otimes \mathrm{End}(V)$ is a fiber of $H^0(tRp_0^\bullet\Omega^*) \otimes \mathrm{End}(V)$ over $(x, x) \in X \times X$. We shall calculate the monodromy of $\tau_{\nabla_{\mathcal{V}}}$. We construct a section s of $H^0(tRp_*^\bullet\Omega^*) \otimes \mathrm{End}(V)$ such that $s(x) = \tau_{\nabla_{\mathcal{V}}}$ and next we find conditions when this setion is flat.

We suppose first that the principal G-bundle $\pi : P \to X$ and the pair $(\mathcal{V}, \{W_i\})$ are trivial. We recall that

$$\tau_{\nabla_{\mathcal{V}}} = \tau^{(0)} + \tau^{(1)} + \dots \tau^{(n)}$$

for some n and we assume that $\tau^{(n)} \neq 0$. Let $K_0(z), \dots, K_n(z) \in \mathcal{O}(X) \otimes \mathrm{End}(V)$ and let $K_i(x) = Id$ for $i = 0, 1, \dots, n$. Then

$$\tau(z) := \tau^{(0)} \circ K_n(z) + \tau^{(1)} \circ K_{n-1}(z) + \dots \tau^{(n)} \circ K_0(z)$$

is a section of $H^0(tR(p_*^\bullet)\Omega^*)$ such that $\tau(x) = \tau_{\nabla_{\mathcal{V}}}$. This section is flat if and only if the functions $K_i(z)$ satisfy the following system of differential equations

$$dK_0(z) + L(z) \circ K_0(z) = 0,$$

$$dK_1(z) + L(z) \circ K_1(z) + C(z) \circ K_0(z) = 0,$$

$$\dots$$

$$dK_n(z) + L(z) \circ K_n(z) + C(z) \circ K_{n-1}(z) = 0.$$

Observe that the functions $K_0(z), K_1(z), \dots, K_n(z)$ coincide with sums of iterated integrals $g_0, g_1, \dots, g_n$ from 4.0.

Definition 4.12.1. Let us set

$$\int_\gamma \tau_{\nabla_{\mathcal{V}}} := \varphi_\gamma(z) + \varphi_\gamma(z) \circ \int_\gamma \varphi_\gamma(t)^{-1} \circ (-C(t)) \circ \varphi_\gamma(t) +$$

$$\varphi_\gamma(z) \circ \int_\gamma \varphi_\gamma(t)^{-1} \circ (-C(t)) \circ \varphi_\gamma(t), \varphi_\gamma(t)^{-1} \circ (-C(t)) \circ \varphi_\gamma(t) + \dots.$$

(γ is a path from x to z, $\varphi_\gamma(z)$ is given by $d\varphi_\gamma(z) + L(z)\varphi_\gamma(z) = 0, \varphi_\gamma(x) = Id$).

Lemma 4.12.2. *Let $\gamma \in \pi_1(X,x)$. Then the monodromy transformation of $\tau_{\nabla_{\mathcal{V}}}$ along γ is given by*

$$\gamma : \tau_{\nabla_{\mathcal{V}}} \to \tau_{\nabla_{\mathcal{V}}} \circ \int_{\gamma} \tau_{\nabla_{\mathcal{V}}}.$$

Proof. It follows from equalities $K_i(z) = g_i(z)$.

Now we consider general case. Let $\mathcal{X}^\bullet = (X,x)^{(\Delta[1],1)}$. We shall calculate cohomology sheaves $\mathcal{H}^i := H^i(tR(p^\bullet)_*\Omega^*)$. (We recall that $\Omega^* = \Omega^*_{\mathcal{X}^\bullet/(x\times X)^\bullet}(\mathcal{P}^{\otimes(\bullet+1)})$.) Next we compute the action of the Gauss-Manin connection on global sections of $\mathcal{H}_i$.

Let us set

$$\mathcal{T} :== Tot(\oplus_{n=0}^{\infty} C^*(\mathcal{U};\Omega^*(\mathcal{P})) \otimes C^*(\mathcal{U};\Omega^0(\mathcal{P}))).$$

We have

$$H^i(\mathcal{T}) = \Gamma(X,\mathcal{H}^i).$$

We define an element $\kappa \in C^0(\mathcal{U};\Omega^0(\mathcal{P})) \otimes \mathrm{End}(V)$ in the following way :

$$\kappa := ((i) \to \tau_{\varphi_i}^{-1} \circ K_i),$$

where $K_i \in \mathcal{O}_X(\mathcal{U}_i) \otimes \mathrm{End}(V)$ and $K_{i_0}(x) = Id_V$. Let us set

$$K := \sum_{n=0}^{\infty} \sigma(g_n) \otimes \ldots \otimes \sigma(g_1) \otimes \kappa \in \mathcal{T} \otimes \mathrm{End}(V).$$

Lemma 4.12.3. *The element K is a gobal section of $\mathcal{H}^0 \otimes \mathrm{End}(V)$ if and only if*

$$B_{ji}K_i + AjiK_i = K_j$$

for all pairs (j,i).

Proof. One verifies that the element K is a cocycle in $\mathcal{T}$ if and only if $B_{ji}K_i + AjiK_i = K_j$.

Observe that a complex

$$\mathcal{T}' := Tot(\oplus_{n=0}^{\infty} C^*(\mathcal{U};\Omega^*(\mathcal{P})^{\otimes n}) \otimes C^*(\mathcal{U};\Omega^*(\mathcal{P})))$$

computes the cohomology groups $H^i(tR\Omega^*_{\mathcal{X}^\bullet}(\mathcal{P}^{\otimes(\bullet+1)}))$. The section s of $\mathcal{H}^i$ (i.e. an element $s \in \mathcal{T}^i$ such that $ds = 0$ in $\mathcal{T}$) is a flat section for the connection $d_1^{0,i} : \mathcal{H}^i \to \Omega^1_{x\times X} \otimes_{\mathcal{O}_{x\times X}} \mathcal{H}^i$ if and only if $ds = 0$ in the complex $\mathcal{T}'$. Hence we get the following result.

Lemma 4.12.4. *The (local) section K is flat over $\cup_{i\in I}\mathcal{U}_i$, $I \subset J$ if and only if*

$$dK_i + \lambda_i K_i + C_i K_i = 0$$

for all n and all $i \in I$.

Observe that K is a flat section of $\mathcal{H}^0 \otimes \mathrm{End}(V)$ if and only κ is a flat section of the principal $GL(V)$-bundle corresponding to $\mathcal{V}$. Let γ be a path from x to z. We denote by $\kappa(z) \in \mathrm{End}(V)$ a value at z of the flat section κ continued along the path γ.

Definition 4.12.5. Let us set

$$\int_\gamma \tau_{\nabla_\mathcal{V}} := \kappa(z).$$

Lemma 4.12.6. *Let $\gamma \in \pi_1(X, x)$. The monodromy transformation of $\tau_{\nabla_\mathcal{V}}$ along γ is given by*

$$\gamma : \tau_{\nabla_\mathcal{V}} \to \tau_{\nabla_\mathcal{V}} \circ \int_\gamma \tau_{\nabla_\mathcal{V}}.$$

The representation $\tau_{\nabla_\mathcal{V}} : V \to \mathcal{H} \otimes V$ of the Hopf algebra $\mathcal{H}$ induces a representation

$$(\tau_{\nabla_\mathcal{V}})^* : (\mathrm{Spec}\,\mathcal{H})(C) \to \mathrm{End}(V)$$

given by

$$(\tau_{\nabla_\mathcal{V}})^*(\sigma) := (\sigma \otimes id_V) \circ \tau,$$

where $\sigma \in (\mathrm{Spec}\,\mathcal{H})(C) = Hom_{C-algebra}(\mathcal{H}, C)$. Let $\Theta_x : \pi_1(X, x) \to (\mathrm{Spec}\,\mathcal{H})(C)$ be the monodromy representation of the principal $(\mathrm{Spec}\,\mathcal{H})(C)$ -bundle $\mathcal{H}^0$.

Lemma 4.12.7. *We have*

$$(\tau_{\nabla_\mathcal{V}}) \circ \Theta_x = \int_\gamma \tau_{\nabla_\mathcal{V}}.$$

Proof. Let $g = \Theta_x(\gamma)$ and let $\tau_{\nabla_\mathcal{V}} = (f_{ij}(-)) \in \mathcal{H} \otimes \mathrm{End}(V)$. We view $f_{ij}(-)$ as functions on $(\mathrm{Spec}\,\mathcal{H})(C)$. After the monodromy transformation along the loop $\gamma \in \pi_1(X, x)$ the element $\tau_{\nabla_\mathcal{V}}$ changes into $(f_{ij}(-g)) = (f_{ij}(-)) \circ (f_{ij}(g))$. Lemma 4.12.6 implies that $(f_{ij}(g)) = \int \gamma \tau_{\nabla_\mathcal{V}}$. It follows from the definition of $(\tau_{\nabla_\mathcal{V}})^*$ that $(\tau_{\nabla_\mathcal{V}})^*(g) = (f_{ij}(g))$. Hence we proved the lemma.

Corollary 4.12.8. *The monodromy homomorphism at x of the connection $\nabla_{\mathcal{V}} : \mathcal{V} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{V}$ is given by $\pi_1(X,x) \ni \gamma \to \int_\gamma \tau_{\nabla_{\mathcal{V}}} \in \mathrm{End}(V)$.*

5. The Classification of variations of mixed Hodge structures.

5.0. Let $VMHS_X(\mathcal{P})$ be a category of variations of mixed Hodge structures $\mathcal{V}$ over X such that $Gr_W\mathcal{V}$ is in $VHS_X(\mathcal{P})$. Let $MHRep(\mathcal{H})$ be a category of mixed Hodge representations of $\mathcal{H}$. Let

$$F : VMHS_X(\mathcal{P}) \to MHRep(\mathcal{H})$$

be a functor which to a variation of mixed Hodge structures $\mathcal{V}$ associates a mixed Hodge representation $\tau_{\nabla_{\mathcal{V}}} : V_x \to \mathcal{H} \otimes V_x$, where V_x is a fiber of $\mathcal{V}$ over $x \in X$. The functor F on morphisms is a restriction to fibers over x. It follows from Lemmas 4.9.3, 4.9.4 and Propositions 4.11.3, 4.11.4 that F is well defined. Our aim is to prove that F is an equivalence of categories.

Theorem 5.0.1. *Let X be a smooth complex projective variety. Then the functor $F : VMHS_X(\mathcal{P}) \to MHRep(\mathcal{H})$ is an equivalence of categories.*

Proof. Let $\mathcal{V}$ and $\mathcal{V}'$ be in $VMHS_X(\mathcal{P})$. First show that

$$F : \mathrm{Hom}_{VMHS_X(\mathcal{P})}(\mathcal{V},\mathcal{V}') \to \mathrm{Hom}_{MHRep(\mathcal{H})}(V_x, V'_x)$$

is bijective. It follows from that the restriction to fibers over x,

$$G : Hom_{VMHS_X(\mathcal{P})}(\mathcal{V},\mathcal{V}') \to Hom_{\pi_1(X,x)}(V_x, V'_x)$$

is bijective. This implies that F is injective. Let $\varphi : V_x \to V'_x$ be a morphism in $MHRep(\mathcal{H})$. It follows from that $\varphi \in \mathrm{Hom}_{\pi_1(X,x)}(V_x, V'_x)$. Hence there is $f : \mathcal{V} \to \mathcal{V}'$ such that $G(f) = \varphi$ and therefore also $F(f) = \varphi$.

It rests to show that any mixed Hodge representation $\tau : V \to \mathcal{H} \otimes V$ is isomorphic to $F(\mathcal{V})$ for a certain variation of mixed Hodge structures $\mathcal{V}$.

Let us set $\mathcal{V}_\tau := (\mathrm{Spec}\,\mathcal{H}^0)(C) \times_{(\mathrm{Spec}\,\mathcal{H})(C)} V$. ($(\mathrm{Spec}\,\mathcal{H})(C)$ acts on V on the right by $(v,g) \to g^{-1}(v)$). Let $\tau' : \mathcal{H}^0 \otimes V \to \mathcal{H} \otimes (\mathcal{H}^0 \otimes V)$ be the diagonal action of $\mathcal{H}$. Let $\iota : \mathcal{H}^0 \otimes V \to \mathcal{H} \otimes (\mathcal{H}^0 \otimes V)$ be given by $\iota(w) = 1 \otimes w$. Then $\mathcal{V}_\tau = (\mathcal{H}^0 \otimes V)^{(\mathrm{Spec}\,\mathcal{H})(C)} =$

$ker(\tau' - \iota)$. This implies that $\mathcal{V}_\tau$ carries a variation of mixed Hodge structures because $\tau' - \iota$ is a morphism of variations of mixed Hodge structures. The monodromy representations at x of the variation $\mathcal{V}_\tau$ is equal to the composition

$$\pi_1(X,x) \stackrel{\Theta_x}{\to} (\mathrm{Spec}\,\mathcal{H}) \stackrel{\tau^*}{\to} \mathrm{End}(V).$$

Let $\tau_{\mathcal{V}_\tau}$ be a cocycle associated to $\mathcal{V}_\tau$. Then the monodromy representation at x of of $\mathcal{V}_\tau$ is equal $(\tau_{\mathcal{V}_\tau})^* \circ \Theta_x$.

6. Classification of algebraic differential equations with regular singular points.

6.0. Let X be a smooth complex algebraic variety, complement of a divisor with normal crossings in a smoth projective variety.

Let $P \to X$ be a holomorphic principal G-bundle equipped with a holomorphic integrable connection. We assume that the image of the monodromy homomorphism at x

$$\theta : \pi_1(X,x) \to G$$

is Zariski dense in G.

Let $DE(X;\theta)_{hol}$ be a category of holomorphic vector bundles $\mathcal{V}$ on X equipped with an increasing filtration $\{W_i\mathcal{V}\}$ by holomorphic sub-vector bundles and with a holomorphic integrable connection $\nabla_{\mathcal{V}}$ compatible with the filtration $\{W_i\mathcal{V}\}$ such that the pair $(Gr_W\mathcal{V}, \nabla_{Gr_W}\mathcal{V})$ ($\nabla_{Gr_W}\mathcal{V}$ is a connection induced by $\nabla_{\mathcal{V}}$ on $Gr_W\mathcal{V}$) is of the form $(P \times_\rho V; \nabla_\rho)$, where $\rho : G \to \mathrm{Aut}V$ is a representation and ∇_ρ is the connection induced on the associated vector bundle by the connection on the principal G-bundle $P \to X$.

We recall from previous sections that given a principal G-bundle $P \to X$ equipped with an integrable connection we have constructed a Hopf algebra $\mathcal{H}$ and a holomorphic bundle $\mathcal{H}^0$ equipped with an integrable holomorphic connnection.

We denote by $Rep(\mathcal{H})$ the category of representations of the Hopf algebra $\mathcal{H}$ in a finite dimensional vector spaces.

Theorem 6.0.1. *The categories $DE(X,\theta)_{hol}$, $Rep(\pi_1(X,x);\theta)$ and $Rep(\mathcal{H})$ are equivalent.*

Proof. It follows from [D1] that the first two categories are equivalent. The equivalence

$$G : DE(X,\theta)_{hol} \to Rep(\pi_1(X,x);\theta)$$

associates to a pair $(\mathcal{V},\nabla)$ a monodromy representation of $\pi_1(X,x)$ in a fiber over x and to a morphism $f : (\mathcal{V},\nabla) \to (\mathcal{V}',\nabla')$ its restriction to fibers over x, $f_x : V_x \to V'_x$.

Let F be a functor from $DE(X;\theta)_{hol}$ to $Rep(\mathcal{H})$, which to a pair $(\mathcal{V},\nabla)$ associates a representation τ_∇ and to a morphism $f : (\mathcal{V},\nabla) \to (\mathcal{V}',\nabla')$ its restriction to fibers over x, $f_x : V_x \to V'_x$. The functor F is well defined by Lemmas 4.1.0.2. and 4.10.3. It is injective on morphisms because the functor G, which on morphisms is also a restriction to fibers over x, is an equivalence of categories. Let $(\mathcal{V},\nabla)$ and $(\mathcal{V}',\nabla')$ be in $DE(X,th)_{hol}$. Let $\alpha \in \mathrm{Hom}_{Rep(\mathcal{H})}(\tau_\nabla,\tau_{\nabla'})$. The monodromy representations of $(\mathcal{V},\nabla)$ and $(\mathcal{V}',\nabla')$ factor through $(\tau_\nabla)^*$ and $(\tau_{\nabla'})^*$ respectively. Hence $\alpha \in \mathrm{Hom}_{Rep(\pi_1(X,x))}(V_x,V'_x)$, where V_x and V'_x are fibers over x of $\mathcal{V}$ and $\mathcal{V}'$ respectively. The fact that the functor G is an equivalance of categories implies that $\alpha = F(f)$ for some $f : (\mathcal{V},\nabla) \to (\mathcal{V}',\nabla')$.

Let $\tau : V \to \mathcal{H} \otimes V$ be a representation of $\mathcal{H}$. Then $\tau^* : (\mathrm{Spec}\,\mathcal{H})(C) \to \mathrm{Aut}(V)$ is a representation of a group $(\mathrm{Spec}\,\mathcal{H})(C)$. The principal $(\mathrm{Spec}\,\mathcal{H})(C)$ -bundle $(\mathrm{Spec}\,\mathcal{H}^0)(C)$ is equipped with the integrable connection, hence the associated vector bundle $\mathcal{V}(\tau) := (\mathrm{Spec}\,\mathcal{H}^0)(C) \times_{\tau^*} V \to X$ is equipped with an integrable connection $\nabla_{\mathcal{V}(\tau)}$ induced by the connection of the principal bundle. The monodromy representation of $\nabla_{\mathcal{V}(\tau)}$ is equal $\tau^* \circ \Theta_x$. On the other hand it is equal $(\tau_{\nabla_{\mathcal{V}(\tau)}})^* \circ \Theta_x$.

6.1. Now we shall prove an algebraic analogue of Theorem 6.0.1.

Let $\bar{X}$ be a smooth proper scheme of finite type over $\mathrm{Spec}\,k$ and let D be a divisor with normal crossings in $\bar{X}$. Let $X := \bar{X} \setminus D$.

We assume that $P \to X$ is an algebraic principal G-bundle equipped with an algebraic integrable regular connection. Let $\sigma : k \hookrightarrow C$ be an embedding. Let us set $X_\sigma := X \times_k C$. We assume that the image of the monodromy homomorphism at x

$$\theta : \pi_1(X_\sigma(C),x) \to G(C)$$

is Zariski dense in G.

Let $DE(X,\theta)_{alg}$ be a category of algebraic vector bundles $\mathcal{V}$ on X equipped with an increasing filtration $\{W_i\mathcal{V}\}$ by algebraic subvector bundles and with an algebraic integrable regular connection $\nabla_{\mathcal{V}}$ compatible with the filtration $\{W_i\mathcal{V}\}$ such that the pair $(Gr_W\mathcal{V}, \nabla_{Gr_W\mathcal{V}})$ is of the form $(P \times_\rho V; \nabla_\rho)$.

We recall from previous sections that given an algebraic principal G-bundle equipped with an algebraic integrable regular connection we have constructed a Hopf k-algebra $\mathcal{H}$ and a bundle $\mathcal{H}^0$ equipped with a regular integrable connection.

Theorem 6.1.1. *The categories $DE(X,\theta)_{alg}$ and $Rep(\mathcal{H})$ are equivalent.*

Proof. First we consider the case $k = C$. Let $h : DE(X,\theta)_{alg} \to DE(X,\theta)_{hol}$ be a functor which to an algebraic object associates a corresponding analytic object. This functor is an equivalence of categories. Hence the functor

$$F : DE(X,\theta)_{alg} \to Rep(\mathcal{H})$$

is an equivalence of categories. This implies that F is an equivalence of categories for any algebraically closed field k contained in C. Let $k \subset C$ and let $\bar{k} \subset C$ be its algebraic closure. We have a commutative diagram of functors

$$\begin{array}{ccc} DE(X\theta)_{alg} & \xrightarrow{F_k} & Rep(\mathcal{H}) \\ \downarrow i_1 & & \downarrow i_2 \\ DE(X \times_k \bar{k}, \theta)_{alg} & \xrightarrow{F_{\bar{k}}} & Rep(\mathcal{H} \otimes_k \bar{k}). \end{array}$$

We shall use the following observation. If a system of linear equations over k has a generic solution in $\bar{k}$, then it has a generic solution in k. This implies that i_1 and i_2 are injective on classes of isomorphisms of objects. The functiors i_1 and i_2 on morphisms behaves like tensoring with $\bar{k}$ over k. Hence the functor F_k is an equivalence of categories.

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The fiber of Hi(trp) over (y,X y) times X can be describe in the following way :

We choose a point X prime in pi-1(X). Hence we get an isomorphism P aprox G. The restriction of Ome y times X n+1 (p ot (n+1)) to y times X times X is ot og. Hence the fiber of Hi(trp) over (y,X) is Hi (t Rga Ome (X y,x)(Del 1 ;y,X) (p ot bullet) ot og.

A. Cosimplicial spaces.

A.1. We review here some definitions from [W]. The category Δ is defined in the following way. The objects of Δ are sequences of integers $\Delta_n = (0, 1, \ldots, n)$. . The morphisms of Δ are monotomic maps $\mu\Delta_n \to \Delta_m$.. Morphisms $\delta^i : \Delta_{n-1} \to \Delta_n \quad n \geq i \geq 0$ given by $\delta^i(j) = j$ if $j < i$ and $\delta^i(j) = j + 1$ if $j \geq i$ are called coface operators. Morphisms $s^j : \Delta_n \to \Delta_{n-1}$ $n-1 \geq j \geq 0$ given by $s^j(k) = k$ if $j \geq k$ and $s^j(k) = k-1$ if $k \geq j+1$ are called codegeneracy operators. A cosimplicial object in a category $\mathcal{C}$ is a covariant functor $X^\bullet : \Delta \to \mathcal{C}$. . We usually denote $X^\bullet(\Delta_n)$ by X^n, $X^\bullet(\delta^i)$ by δ^i and $X^\bullet(s^j)$ by s^j.

A.1. Let $K^{*,\bullet}$ be a bicomplex with commuting differentials $\partial^{i,j} : K^{i,j} \to K^{i+1,j}$ and $\delta^{i,j} : K^{i,j} \to K^{i,j-1}$. We define the total complex of $K^{*,\bullet}$ in the following way:

$$(TotK^{*,\bullet})_m := \oplus_{i-j=m} K^{i,j},$$

$$d_m : (TotK^{*,\bullet})_m \to (TotK^{*,\bullet})_{m-1}$$

and

$$d_{m|K^{i,j}} = \partial^{i,j} + (-1)^i \delta^{i,j}.$$

Let $X^\bullet$ be a cosimplicial space. A sheaf on $X^\bullet$ consists of sheaves F_n on X^n together with maps $F_m \to \alpha_* F_n$ for any $\Delta_n \to \Delta_m$ in Δ satisfying the obvious compatibility conditions. If $F_\bullet$ is a sheaf on $X^\bullet$ with values in an abelian category $\mathcal{C}$ then the global section functor on $X^\bullet$,

$$\Gamma(F_\bullet; X^\bullet) : n \to \Gamma(F_n; X^n)$$

is a simplicial object in $\mathcal{C}$. The obvious functor

(simplicial objects in $\mathcal{C}$)$\to$ (complexes in $\mathcal{C}$) associates to $\Gamma(F_\bullet; X^\bullet)$ and hence also to $F_\bullet$, a complex which we shall denote also by $\Gamma(F_\bullet; X^\bullet)$.

A.2. Let $X^\bullet$ be a cosimplicial space. A category of sheaves of abelian groups on $X^\bullet$ is an abelian category which we denote by $Ab(X^\bullet)$. If $F_\bullet$ is a sheaf of abelian groups on $X^\bullet$, then $F_\bullet$ has a right resolution $K^*_\bullet$ in $Ab(X^\bullet)$ such that $H^r(X^q; K^p_q) = 0$ for $r > 0$. One can take a canonical Godement resolution. The resolution $K^*_\bullet$, after applying the functor of global sections leads to a bicomplex $\Gamma(K^*_\bullet; X^\bullet)$. One defines

$$H^n(X^\bullet; F_\bullet) := H^n(Tot\Gamma(K^*_\bullet; X^\bullet)).$$

Let $F^*_\bullet$ be a complex of sheaves of abelian groups on $X^\bullet$. One shows that there is a quasi-isomorphism $F^*_\bullet \to K^*_\bullet$ such that $H^r(X^q; K^p_q) = 0$ for $r > 0$. One can take for each pair (n, m) a canonical Godement resolution of F^m_n, $C^*(X^n; F^m_n)$. Then $C^p(X^n; F^q_n))_{p,q)}$ is a bicomplex of sheaves on X^n and K^*_n is its total comples. We define (hyper-) cohomology of $X^\bullet$ with coefficients in $F^*_\bullet$ in the following way:

$$H^n(X^\bullet; F^*_\bullet) := H^n(Tot\Gamma(K^*_\bullet; X^\bullet)).$$

A.3. Let $A^*_\bullet$ be a complex of sheaves of k-vector spaces on a cosimplicial space $X^\bullet$. We say that $A^*_\bullet$ is a sheaf of commutative differential graded k-algebras on $X^\bullet$ if for each n, A^*_n is a sheaf of commutative differential graded k-algebras on X^n and if the structure maps are morphisms of k-algebras.

Example. $\Omega^*_{X^\bullet}$ (in degree n, on X^n, we have the De Rham complex $\Omega^*_{X^n}$) is a sheaf of commutative differential graded k-algebras on $X^\bullet$.

It follows from [N] that for any sheaf $A^*_\bullet$ of cdg k-algebras on $X^\bullet$, there is a sheaf $I^*_\bullet$ of cdg k-algebras on $X^\bullet$, and a quasi-isomorphism, a morphism of k-algebras $A^*_\bullet \to I^*_\bullet$ such that I^*_n is a complex of flasque sheaves for each n. Then $(n \to \Gamma(I^*_n, X^n))$ is a simplicial object in the categoroy of commutative graded differential k-algebras. k-algebras. We shall describe how to introduce the product in the cohomology of the total complex.

Let $A^{*,\bullet} = \{n \to A^{*,n}\}_n$ be a simplicial object in the category of cdg k-algebras with face maps δ_i and degeneracy maps s_j. We shall define a shuffle product in the total complex $TotA$ by the following formula:

if $x \in A^{p_1,q_1}$ and $y \in A^{p_2,q_2}$ then

$$x * y := \sum_{(q_1,q_2)-\text{shufles } \sigma^{-1}} (s_{\sigma(q_2+q_1)} \circ \ldots s_{\sigma(q_1+1)}(x)) \cdot (s_{\sigma(q_1)} \circ \ldots s_{\sigma(1)})(y),$$

$$\pi_1(X, x) \overset{\Theta_x}{\to} (\operatorname{Spec} \mathcal{H}) \overset{\tau^*}{\to} \operatorname{End}(V).$$

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