Classification of variations of mixed Hodge structures I

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0. Introduction.

0.1. The purpose of this note is to study a classification of variation of mixed Hodge structures over a given complex variety \( X \). The starting point is the following well known result.

Let \( \tilde{X} \rightarrow X \) be a universal covering space over \( X \). The fundamental group \( \pi_1(X, x) \) acts on \( \tilde{X} \). Let \( V \) be a finite dimensional vector space over \( k \) and let \( \rho : \pi_1(X, x) \rightarrow \text{Aut}(X) \) be a representation. Then \( \tilde{X} \times_\rho V \rightarrow X \) is a local system of \( k \)-vector spaces over \( X \).

**Theorem 0.1.1.** The category of local systems of finite dimensional \( k \)-vector spaces over \( X \) and the category of representations of \( \pi_1(X, x) \) in finite dimensional \( k \)-vector spaces are equivalent.

Let \( G \) be a complex affine algebraic group. Let \( \pi : P \rightarrow X \) be a principal \( G \)-bundle equipped with an integrable connection \( \omega \). We assume that the image of the monodromy homomorphism \( \Theta : \pi_1(X, x) \rightarrow G \) is Zariski dense in \( G \). Let \( \mathcal{O}[G] \) be the ring of regular functions on \( G \). The group \( G \) acts on \( \mathcal{O}[G] \) on the left by \((g \cdot f)(t) := f(g^{-1}t)\). The associated vector bundle is the bundle \( \pi_*\mathcal{O}_P \rightarrow X \). We equip it with a connection \( \nabla_\omega \) induced by the connection \( \omega \). Let us denote the vector bundle \( \pi_*\mathcal{O}_P \rightarrow X \) by \( \mathcal{P} \). We assume that \( \mathcal{P} \) carries a variation of Hodge structures of weight 0 (the connection is \( \nabla_\omega \)) and that \( \mathcal{O}[G] \) carries a Hodge structure of weight 0. We assume that these Hodge structures are compatible with the multiplicative structures on \( \mathcal{P} \) and on \( \mathcal{O}[G] \), with the Hopf algebra structure on \( \mathcal{O}[G] \) and with the (co-) action of \( \mathcal{O}[G] \) on \( \mathcal{P} \).

**Definition 0.1.2.** Let \( V \) be a finite direct sum of Hodge structures \( V_i \) of weight \( n_i \). We say that

\[
\rho : V \rightarrow \mathcal{O}[G] \otimes V
\]

is a Hodge representation if \( \rho \) is a morphism of Hodge structures and

\[
(id_{\mathcal{O}[G]} \otimes \rho) \circ \rho = (\mu \otimes id_{V}) \circ \rho,
\]

where \( \mu \) is induced by a multiplication \( G \times G \rightarrow G \).

The group \( G \) acts on \( V \) \((G \ni g \rightarrow \rho(g) \in \text{End}(V))\). We form an associated vector bundle \( \mathcal{V}_\rho := (P \times_G V \rightarrow X) \). We shall see that \( \mathcal{V}_\rho \) carries a variation of Hodge structures.
Consider the diagonal action of $G$ on $\mathcal{P} \otimes V^*$. Then $(\mathcal{P} \otimes V^*)^G$ is the bundle $\mathcal{V}_\rho^*$. The bundles $\mathcal{P} \otimes V^*$ and $\mathcal{P} \otimes V^* \otimes \mathcal{O}[G]$ carry variations of Hodge structures. The action of $G$ on $\mathcal{P}$ and $V^*$ are given by Hodge representations (morphisms of variations of Hodge structures) $\mathcal{P} \to \mathcal{O}[G] \otimes \mathcal{P}$ and $V^* \to \mathcal{O}[G] \otimes V^*$. Hence the diagonal action of $G$ on $\mathcal{P} \otimes V^*$ is given by a Hodge representation $\mathcal{P} \otimes V^* \to \mathcal{O}[G] \otimes \mathcal{P} \otimes V^*$. Observe that $(\mathcal{P} \otimes V^*)^G = \{ x \in \mathcal{P} \otimes V^* \mid \tau(x) = 1 \otimes x \}$. This implies that the vector bundle $\mathcal{V}_\rho^* = (\mathcal{P} \otimes V^*)^G$ carries a variation of Hodge structures. Hence also $\mathcal{V}_\rho$ carries a variation of Hodge structures.

We denote by $HRep(\mathcal{O}[G])$ the category of Hodge representations of $\mathcal{O}[G]$. We denote by $V_{\mathcal{HRep}(\mathcal{O}[G])}^\mathcal{P}$ the category of variations of Hodge structures of the form $\mathcal{V}_\rho (\rho \in HRep(\mathcal{O}[G]))$.

**Theorem 0.1.3.** Assume that the Hodge structures $\mathcal{O}[G]$, $V$ and the variation of Hodge structures $\mathcal{P}$ carry polarizations and the structure morphisms are compatible with polarizations. Then the categories $HRep(\mathcal{O}[G])$ and $V_{\mathcal{HRep}(\mathcal{O}[G])}^\mathcal{P}$ are equivalent.

**0.2.** We denote by $VMHS_\mathcal{P}$ the category of variations of mixed Hodge structures $\mathcal{V}$ over $X$ such that $Gr^W_i \mathcal{V} \in V_{\mathcal{HRep}(\mathcal{O}[G])}^\mathcal{P}$ for each $i$. Our aim is to classify such variations. First we construct a universal variation of mixed Hodge structures.

The inclusion of simplicial sets $\partial \Delta[1] \hookrightarrow \Delta[1]$ induces a morphism of cosimplicial spaces $p^* : X^{\Delta[1]} \to X^{\partial \Delta[1]}$. Let $y \in X$. Let us set $(X, y)^{\Delta[1], 1} := p^* -1(\{ y \times X \}^\bullet)$, where $(y \times X)^\bullet$ is a constant cosimplicial space equal $y \times X$ in each degree. The sheaf $\mathcal{P}$ is a sheaf of $\mathcal{O}_X$-algebras and the connection $\nabla : \mathcal{P} \to \Omega^1_X \otimes \mathcal{O}_X \mathcal{P}$ is multiplicative. Hence the relative twisted De Rham complexes form a complex of sheaves $\Omega^* := \Omega_{(X, y)^{\Delta[1], 1}} / (y \times X)^\bullet(\mathcal{P}^{\otimes (\bullet + 1)})$ on $(X, y)^{\Delta[1], 1}$. Applying a functor $R\pi^*$ we get a bicomplex of $\mathcal{O}_X$-modules on $y \times X$. Let $t$ be a functor which to a bicomplex associates its total complex.

**Theorem 0.2.1.** We have

- **i)** The cohomology sheaves $H^i(tR\pi^* \Omega^*)$ carry a variation of mixed Hodge structures.
- **ii)** $H^0(tR\pi^* \Omega^*)$ is a sheaf of $\mathcal{O}_X$-algebras and the mixed Hodge structure is compatible with multiplication;
iii) Let $\mathcal{H}$ be a fiber of $H^0(tRp_*\Omega^*)$ over $(y,y)$. Then $\mathcal{H}$ is a Hopf algebra, $\mathcal{H}$ (co-)acts on $H^0(tRp_*\Omega^*)$ and the structure morphisms are morphisms of mixed Hodge structures.

0.3. Let $V \in VMHS_p$ and let $V$ be a fiber of $V$ over $y \in X$. Our aim is to construct a mixed Hodge representation

$$\tau_V : V \to \mathcal{H} \otimes V.$$ 

The following observation is crucial in the construction of $\tau_V$. Let $L(z)$ and $C(z)$ be in $\Omega^1(X) \otimes \text{End}(V)$. Assume that $\varphi(z)$ satisfies an equation 

$$d\varphi(z) + L(z)\varphi(z) = 0, \quad \varphi(y) = Id_V.$$ 

Then the sum of iterated integrals

$$F(z) := \varphi(z)(Id + \int \varphi(z)^{-1} \circ (-C(z)) \circ \varphi(z) + \int \varphi(z)^{-1} \circ (-C(z)) \circ \varphi(z), \varphi(z)^{-1} \circ (-C(z)) \circ \varphi(z) + \ldots)$$ 

satisfies an equation 

$$dF(z) + (L(z) + C(z))F(z) = 0.$$ 

We assume for simplicity that the bundle $V$ and the sub-bundles $W_iV$ are trivial. The bundle $Gr_W V$ is obtained from a Hodge representation $\tau_\varphi \in \mathcal{O}[G] \otimes \text{End}(V)$. It is equipped with a connection form $L(z) \in \Omega^1(X) \otimes \text{End}(V)$. The bundle $V$ is equipped with a connection form $L(z) + C(z)$ and $C(z)$ vanishes when passing to the associated graded bundle.

Let us set

0.3.1. 

$$\tau_V := \tau_\varphi(g_0)^{-1} + \tau_\varphi(g_1)^{-1} \circ (-C(z_1)) \circ \tau_\varphi(g_1) \circ \tau_\varphi(g_0)^{-1} +$$

$$\tau_\varphi(g_2)^{-1} \circ (-C(z_2)) \circ \tau_\varphi(g_2) \circ \tau_\varphi(g_1)^{-1} \circ (-C(z_1)) \circ \tau_\varphi(g_1) \circ \tau_\varphi(g_0)^{-1} + \ldots$$

$$\in \oplus_{n=0}^{\infty} \Omega^n(y \times X^n \times y) \otimes \mathcal{O}[G]^{\otimes (n+1)} \otimes \text{End}(V).$$

**Theorem 0.3.2.** We have

i) $\tau_V$ is a cocycle (the corresponding cohomology class in $\mathcal{H} \otimes \text{End}(V)$ we denote also by $\tau_V$).

ii) $\tau_V : V \to \mathcal{H} \otimes V$ is a mixed Hodge representation.
iii) If $\mathcal{V}$ and $\mathcal{V}'$ are isomorphic variations of mixed Hodge structures then $\tau_{\mathcal{V}} = \tau_{\mathcal{V}'} \in \mathcal{H} \otimes \text{End}(V)$.

We denote by $\text{MHRep}(\mathcal{H})$ the category of mixed Hodge representations of $\mathcal{H}$. We define a functor $F: VMHS_P \to \text{MHRep}(\mathcal{H})$ by $F(\mathcal{V}) := \tau_{\mathcal{V}}$. The functor $F$ on morphisms is a restriction to fibers over $y$.

**Theorem 0.3.3.** Let $X$ be a smooth complex projective variety. Then the functor

$$F : VMHS_P \to \text{MHRep}(\mathcal{H})$$

is an equivalence of categories.

Let $X$ be a complement of a divisor with normal crossings in a smooth complex projective variety. Let $DE(X, \mathcal{P})_{alg}$ be the category of algebraic vector bundles $\mathcal{V}$ over $X$ equipped with a filtration $\{W_i \mathcal{V}\}$ and a regular integrable connection compatible with the filtration such that the associated graded vector bundle equipped with the induced connection is isomorphic to $P \times_\varphi V$ for some representation $\varphi : G \to \text{Aut}(V)$.

**Theorem 0.3.4.** The functor $F : DE(X, \mathcal{P})_{alg} \to \text{Rep}(\mathcal{H})$ is an equivalence of categories.
§1. Hodge representations.

1.0. Let $G$ be a reductive algebraic group over a field of complex numbers $C$ and let $\mathcal{O}[G]$ be a Hopf algebra of regular functions on $G$ with structure morphisms $\mu : \mathcal{O}[G] \to \mathcal{O}[G] \otimes \mathcal{O}[G]$, $\iota : \mathcal{O}[G] \to \mathcal{O}[G]$ and $\varepsilon : \mathcal{O}[G] \to C$. We assume that $\mathcal{O}[G]$ carries a Hodge structure of weight 0 and that the structure morphisms and the multiplication $m : \mathcal{O}[G] \otimes \mathcal{O}[G] \to \mathcal{O}[G]$ are morphisms of Hodge structures.

Let $V$ be vector space over $C$. We say that $\mathcal{O}[G]$ acts on $V$ if there is a linear map $\tau_V : V \to \mathcal{O}[G] \otimes V$ such that

$$(\mu \otimes \text{id}_V) \circ \tau_V = (\text{id}_{\mathcal{O}[G]} \otimes \tau_V) \circ \tau_V \quad \text{and} \quad (\varepsilon \otimes \text{id}_V) \circ \tau_V = \text{id}_V.$$ 

We denote also by $\tau_V : G \to \text{Aut} V$ an action of $G$ on $V$ induced by an action $\tau_V$ of $\mathcal{O}[G]$ on $V$. Let $g \in G$. Then the automorphism $\tau_V(g)$ is the composition $e_{g^{-1}} \circ \tau_V$, where $e_{g^{-1}} : \mathcal{O}[G] \to C$ is the evaluation map at $g^{-1}$.

Assume that $V$ carries a Hodge structures of weight $n$. We say that $\tau : V \to \mathcal{O}[G] \otimes V$ is an irreducible Hodge representation if $\tau$ is a morphism of Hodge structures and if the representation $\tau : G \to \text{Aut} V$ is irreducible. A direct sum of irreducible Hodge representations is called a Hodge representation of $\mathcal{O}[G]$. We denote by $\text{HRep}(\mathcal{O}[G])$ the category of Hodge representations of $\mathcal{O}[G]$.

Observe that any Hodge representation $\tau : V \to \mathcal{O}[G] \otimes V$ determines an element of Hodge type $(0, 0)$ in $\mathcal{O}[G] \otimes V^* \otimes V = \mathcal{O}[G] \otimes \text{End}(V)$. This element we shall also denote by $\tau$.

1.1. Let $X$ be a Zariski open in a smooth complex compact analytic variety. Let $\pi : P \to X$ be a principal $G$-bundle equipped with the integrable connection. Let $\Theta : \pi_1(X, x) \to G$ be the monodromy representation at $x \in X$. Assume that the image of $\Theta$ is Zariski dense in $G$. Assume that $\mathcal{P} := \pi_* \mathcal{O}_P \to X$ equipped with the connection deduced from the connection on $\pi : P \to X$ carries a variation of Hodge structures of weight 0 compatible with the action of $\mathcal{O}[G]$ equipped with a Hodge structure by 1.0. The choice of a point $x' \in \pi^{-1}(x)$ identifies $\mathcal{P}_x$, the fiber of $\mathcal{P}$ over $x$, with $\mathcal{O}[G]$.

We identify a vector bundle with its sheaf of sections.
Lemma 1.1.1. Let $G$ acts on $\mathcal{O}[G]$ on the left by $(gf)(t) := f(g^{-1}t)$. Then the associated vector bundle $P \times_G \mathcal{O}[G] \to X$ is $\pi_*\mathcal{O}_P \to X$.

Proof. One cheks that the transition functions are the same for both bundles.

Let $\tau : V \to \mathcal{O}[G] \otimes V$ be a Hodge representation. Hence the group $G$ acts on $V$ by $\tau : G \to \text{Aut} V$ and we can form an associated vector bundle $\mathcal{V}_\tau := (P \times_G V \to X)$. We shall see that $\mathcal{V}_\tau$ carries a variation of Hodge structures.

The group $G$ acts on $P$ on the right hence it acts on $\mathcal{P}$ on the left. $G$ acts also on $V^*$ on the left by $(gf)(v) := f(g^{-1}v)$. Hence $G$ acts diagonally on $\mathcal{P} \otimes V^*$.

Lemma 1.1.2. The vector bundle $\mathcal{V}_\tau$ is equal to the dual of the vector bundle $(\mathcal{P} \otimes V^*)^G$.

Proof. We can suppose that the Hodge representation $\tau$ is irreducible. Let $G \times G$ acts on $\mathcal{O}[G]$ by $((h,g)f)(t) := f(h^{-1}tg)$ and on $V^* \otimes V$ by $(h,g)(f(-) \otimes v) := f(h^{-1}-) \otimes g(v)$.

Then $V^* \otimes V$ is included in $\mathcal{O}[G]$ by a $G \times G$-equivariant map. Observe that for the diagonal action of $G$ on $V^* \otimes V \otimes V^*$ given by $g(f_1 \otimes v \otimes f_2) := f_1 \otimes g(v) \otimes f_2 \circ g^{-1}$ we have $((V^* \otimes V) \otimes V^*)^G = V^* \otimes (V \otimes V^*)^G = V^* \otimes C = V^*$. Hence we get $(\mathcal{O}[G] \otimes V^*)^G = V^*$ and $V^*$ is equipped with the action of $G$ deduced from the action of $G$ on $\mathcal{O}[G]$ given by $h(f)(t) = f(h^{-1}t)$. Therefore the transition functions of the bundles $\mathcal{V}_\tau^*$ and $(\mathcal{P} \otimes V^*)^G$ are the same.

Lemma 1.1.3. If $\tau_V$ and $\tau_W$ are Hodge representations then $\tau_{V \otimes W}$ and $\tau_{V^*}$ are Hodge representations.

Proof. Observe that $\tau_V \otimes \tau_W : V \otimes W \to (\mathcal{O}[G] \otimes V) \otimes (\mathcal{O}[G] \otimes W)$, $s : (\mathcal{O}[G] \otimes V) \otimes (\mathcal{O}[G] \otimes W) \to \mathcal{O}[G] \otimes \mathcal{O}[G] \otimes V \otimes W$ given by $s((f \otimes v) \otimes (g \otimes w)) = f \otimes g \otimes v \otimes w$ and $m \otimes id_{V \otimes W}$ are morphisms of Hodge structures. Hence their composition $\tau_{V \otimes W} := (m \otimes id_{V \otimes W}) \circ s \circ (\tau_V \otimes \tau_W)$ is also a morphism of mixed Hodge structures.

We shall show that $\tau_{V^*}$ is a Hodge representation. Let $\tau_V : V \to \mathcal{O}[G] \otimes V$. Then there is a finite dimensional Hodge structure $\mathcal{O}' \subset \mathcal{O}[G]$ such that $\tau_V$ factors through $V \to \mathcal{O}' \otimes V$.

Passing to dual vector spaces we get morphisms of Hodge structures $(\mathcal{O}')^* \otimes V^* \to V^*$. Hence $\tau_{V^*} : V^* \to (\mathcal{O}')^{**} \otimes V^* = \mathcal{O}' \otimes V^* \subset \mathcal{O}[G] \otimes V^*$ is also a morphism of Hodge structures.
Lemma 1.1.4. The vector bundle $V$ carries a variation of Hodge structures.

Proof. The actions of $G$ on $P$ and on $V^*$ are given by a morphism of variations of Hodge structures $P \rightarrow \mathcal{O}[G] \otimes P$ and a Hodge representation $\tau_{V^*} : V^* \rightarrow \mathcal{O}[G] \otimes V^*$. Hence a diagonal action of $G$ on $P \otimes V^*$ is given by a morphism of variations of Hodge structures $P \rightarrow \mathcal{O}[G] \otimes P \otimes V^*$. Let $\iota : P \otimes V^* \rightarrow \mathcal{O}[G] \otimes P \otimes V^*$ be given by $\iota(x) = 1 \otimes x$, where $x \in P \otimes V^*$. Then $\iota$ is a morphism of variations of Hodge structures. Observe that $(P \otimes V^*)^G = \ker(\rho - \iota)$. This implies that the bundle $(P \otimes V^*)^G$ and hence also the bundle $V$ carry a variation of Hodge structures.

1.1.5. Let $x \in X$. Then the choice of a point $x' \in \pi^{-1}(x)$ defines a bijection of $G$ onto $P_x := \pi^{-1}(x)$. Let us denote by $P_x$ the fiber of $P$ over $x$. Then $P_x = \mathcal{O}[P_x]$. We assume that there is $x \in X$ and $x' \in \pi^{-1}(x)$ such that the induced morphism $P_x \rightarrow \mathcal{O}[G]$ is an isomorphism of Hodge structures.

Let $\tau : V \rightarrow \mathcal{O}[G] \otimes V$ be a Hodge representation. Then the fiber of the variation of Hodge structures $\mathcal{V}_\tau$ over $x$ is $V$.

1.2. Polarization (see [Sch] p.217 and 220).

We assume that the Hodge structures $\mathcal{O}[G]$ and $V$ and the variation of Hodge structures $P$ carry polarizations and the structure morphisms, multiplication, actions of $\mathcal{O}[G]$ are compatible with polarizations. Then $(P \otimes V^*)^G$ also carries a polarization because it is a kernel of a morphism between polarized variations of Hodge structures. Hence $\mathcal{V}_\tau$ is a polarized variation of Hodge structures.

Further in this section a Hodge structure and a variation of Hodge structures are always polarized.

1.3. Let $\Theta : \pi_1(X, x) \rightarrow G$ be a homomorphism from 1.1. We denote by $VHS_{\Theta}$ the category of variations of Hodge structures $\mathcal{V}$ over $X$ such that the monodromy homomorphism at $x$

$$\pi_1(X, x) \rightarrow \text{Aut}V_x$$

is a composition

$$\tau_{\mathcal{V}} \circ \Theta : \pi_1(X, x) \rightarrow G \rightarrow \text{Aut}V_x,$$

where $\tau_{\mathcal{V}} : V_x \rightarrow \mathcal{O}[G] \otimes V_x$ is a Hodge representation.
Theorem 1.3.1. The categories $HRep(\mathcal{O}[G])$ and $VHS_\Theta$ are equivalent.

**Proof.** The functor $\mathcal{V} : HRep(\mathcal{O}[G]) \to VHS_\Theta$ associates to a Hodge representation $\tau$ a variation of Hodge structures $\mathcal{V}_\tau$. Let $\mathcal{V} \in VHS_\Theta$ and let $\tau : V_x \to \mathcal{O}[G] \otimes V_x$ be a Hodge representation, which appears in the factorization of the monodromy homomorphism of $\mathcal{V}$ at the point $x$. The variation of Hodge structures $\mathcal{V}_\tau$ has the monodromy homomorphism at $x$ given by $\tau \circ \Theta : \pi_1(X,v) \to G \to \text{Aut}(V_x)$. Hence the monodromy homomorphisms of $\mathcal{V}$ and $\mathcal{V}_\tau$ are equal. It follows from [Sch] Theorem 7.24 that the variations of Hodge structures $\mathcal{V}$ and $\mathcal{V}_\tau$ are isomorphic.

It rests to show that the functor $\mathcal{V}$ on morphisms

$$\text{Hom}_{HRep(\mathcal{O}[G])}(\tau, \tau') \to \text{Hom}_{VHS_\Theta}(\mathcal{V}_\tau, \mathcal{V}_{\tau'})$$

is bijective. It is clear that it is injective. Let $f : \mathcal{V}_\tau \to \mathcal{V}_{\tau'}$ be a morphism of variation of Hodge structures. Restricting $f$ to the fiber over $x$ we get $f_x : \text{Rep}_{\pi_1(X,x)}(V,V')$. The monodromy representations of $\pi_1(X,x)$ on $V$ and $V'$ factor through $\Theta : \pi_1(X,x) \to G$ because $\mathcal{V}_\tau$ and $\mathcal{V}_{\tau'}$ are associated vector bundles equipped with connections induced from the principal $G$-bundle $P \to X$. The image of $\Theta$ is Zariski dense in $G$. This implies that the diagram

$$
\begin{array}{ccc}
V & \xrightarrow{\tau} & \mathcal{O}[G] \otimes V \\
\downarrow f_x & & \downarrow \text{id}_{\mathcal{O}[G]} \otimes f_x \\
V' & \xrightarrow{\tau'} & \mathcal{O}[G] \otimes V'
\end{array}
$$

commutes. Let $\mathcal{V}(f_x) : \mathcal{V}_\tau \to \mathcal{V}_{\tau'}$ be a map induced by $f_x$. $\mathcal{V}(f_x)$ is a flat section of the variation of Hodge structures $\text{Hom}(\mathcal{V}_\tau, \mathcal{V}_{\tau'})$ and it is of Hodge type $(0,0)$ at $x$. It follows from [Sch] Corollary 7.23 that $\mathcal{V}(f_x)$ is a morphism of variations of Hodge structures. $\mathcal{V}(f_x)$ and $f$ coincide at $x$, hence $\mathcal{V}(f_x) = f$ because two flat sections equal at one point are equal everywhere.
§2. Chain complexes.

2.0. Let $X$ be a topological space. The inclusion of simplicial sets $\partial \Delta[1] \to \Delta[1]$ induces a morphism of cosimplicial spaces $p^\bullet : X^\Delta[1] \to X^\partial \Delta[1]$. Let $(x_2, x_1) \in X \times X$. We consider $(x_2, x_1)$ as a constant cosimplicial space equal $(x_2, x_1)$ in each degree. Then $p^{\bullet-1}(x_1, x_2)$ is a cosimplicial space equal $x_2 \times X^n \times x_1$ in degree $n$.

If $Y$ is a smooth complex variety, we denote by $\Omega^*(Y)$ the De Rham complex of smooth complex valued differential forms on the smooth complex variety $Y$.

Let $X$ be a smooth complex variety. Then $\bigoplus_{n=0}^\infty \Omega^*(x_2 \times X^n \times x_1)$ is a bicomplex with two commuting differentials. Let $T_1 := Tot(\bigoplus_{n=0}^\infty \Omega^*(x_2 \times X^n \times x_1))$ be the total complex of the bicomplex $\bigoplus_{n=0}^\infty \Omega^*(x_2 \times X^n \times x_1)$ (see [W1]).

2.1. Let $\partial : \Omega^*(X)^{\otimes n} \to \Omega(X)^{\otimes n}$ be the exterior differentiation. Let us define

$$\delta_i : \Omega^*(X)^{\otimes n} \to \Omega^*(X)^{\otimes (n-1)}$$

in the following way:

$$\delta_i(\omega_n \otimes \ldots \otimes \omega_1) := \omega_n \otimes \ldots \otimes \omega_{i+1} \wedge \omega_i \otimes \ldots \otimes \omega_1 \text{ for } i \neq 0, 1;$$

$$\delta_0(\omega_n \otimes \ldots \otimes \omega_1) := \omega_n \otimes \ldots \otimes \omega_2 \varepsilon_1(\omega_1);$$

$$\delta_n(\omega_n \otimes \ldots \otimes \omega_1) := \varepsilon_2(\omega_n)\omega_{n-1} \otimes \ldots \otimes \omega_1$$

where $\varepsilon_i(\omega) = e(x_i)$ if $\omega \in \Omega^0(X)$ and $\varepsilon_i(\omega) = 0$ if $\deg \omega > 0$. Let us set

$$\delta := \sum_{i=0}^n (-1)^i \delta_i : \Omega^*(X)^{\otimes n} \to \Omega(X)^{\otimes (n-1)}.$$

Then we have $\partial^2 = \delta^2 = \partial \delta - \delta \partial = 0$. Hence $\bigoplus_{n=0}^\infty \Omega^*(X)^{\otimes n}$ is a double complex. We define a total complex $T_2$ such that in degree $k$

$$(T_2)_k = Tot(\bigoplus_{n=0}^\infty \Omega^*(X)^{\otimes n})_k := \bigoplus_{k_1 + \ldots + k_i - i = k} \Omega^{k_1}(X) \otimes \ldots \otimes \Omega^{k_i}(X).$$

If $\omega \in \Omega^{k_1}(X) \otimes \ldots \otimes \Omega^{k_i}(X)$ then $d(\omega) := \partial \omega + (-1)^{k_1 + \ldots + k_i} \delta \omega$, where $k_i = \deg \omega_i$.

The obvious map $T_2 \to T_1$ is a quasi-isomorphism.
2.2. We shall compare the complex \((\text{Tot}(\bigoplus_{n=0}^{\infty} \Omega^*(X)^{\otimes n}, d))\) with the bar construction on \(\Omega^*(X)\), because certain constructions are more familiar in terms of the bar construction. We set
\[
B^{-n} = B^{-n}_{x_2, x_1} = B^{-n}_{x_2, x_1}(\Omega^*(X)) := \Omega^*(X)^{\otimes n}.
\]
The differential \(\partial' : B^{-n} \rightarrow B^{-n}\) is defined by
\[
\partial'(\omega_1 \otimes \ldots \otimes \omega_n) := \sum_{i=1}^{n} (-1)^{k_1 + \cdots + k_{i-1} + i} \omega_1 \otimes \cdots \partial \omega_i \otimes \cdots \otimes \omega_n.
\]
The differential \(\delta' : B^{-n} \rightarrow B^{-n+1}\) is defined by
\[
\delta'(\omega_1 \otimes \ldots \otimes \omega_n) := \varepsilon_2(\omega_1) \omega_2 \otimes \cdots \otimes \omega_n + \sum_{i=1}^{n-1} (-1)^{k_1 + \cdots + k_i + i} \omega_1 \otimes \cdots \otimes \omega_i \wedge \omega_{i+1} \otimes \cdots \otimes \omega_n + (-1)^{k_1 + \cdots + k_{n-1} + k_n} \omega_1 \otimes \cdots \otimes \omega_{n-1} \varepsilon_1(\omega_n).
\]
The bar construction on \(\Omega^*(X)\), \(B^\bullet_{x_2, x_1} = (B^\bullet_{x_2, x_1}(\Omega^*(X)), d')\) is the total complex of the bicomplex \(\bigoplus_{n=0}^{\infty} B^{-n}\) equipped with the differential \(d' := \partial' + \delta'\).

2.3. Let \(x_1, x_2, x_3 \in X\). We recall that there are chain maps
\[
M_{x_1 x_2 x_1} : B^\bullet_{x_3 x_1} \rightarrow B^\bullet_{x_3 x_2} \otimes B^\bullet_{x_2 x_1}
\]
given by
\[
M(\omega_1 \otimes \ldots \otimes \omega_n) = \sum_{i=0}^{n} (\omega_1 \otimes \ldots \otimes \omega_i) \otimes (\omega_{i+1} \otimes \ldots \otimes \omega_n)
\]
and
\[
\iota_{x_2 x_1} : B^\bullet_{x_2 x_1} \rightarrow B^\bullet_{x_1 x_2}
\]
given by
\[
\iota_{x_2 x_1}(\omega_1 \otimes \ldots \otimes \omega_n) := (-1)^{c(n) + 1 + \left(\frac{1}{2} + \frac{(-1)^n}{2}\right)(k_1 + \cdots + k_n)} + \sum_{i<j} k_i k_j \omega_n \otimes \ldots \otimes \omega_1
\]
where \(c(n) = 0\) if \(n \equiv 1, 2 \mod 4\) and \(c(n) = 1\) if \(n \equiv 3, 4 \mod 4\). Observe that \(c(n) + 1 + \left(\frac{1}{2} + \frac{(-1)^n}{2}\right)(k_1 + \cdots + k_n) + \sum_{i<j} k_i k_j = \frac{n(n+1)}{2} + (n+1)(k_1 + \cdots + k_n) + \sum_{i<j} k_i k_j\).

A \((p, q)\)-shuffle is a permutation \(\sigma\) of \((1, 2, \ldots, p + q)\) such that \(\sigma^{-1}(1) < \sigma^{-1}(2) < \ldots < \sigma^{-1}(p)\) and \(\sigma^{-1}(p + 1) < \sigma^{-1}(p + 2) < \ldots < \sigma^{-1}(p + q)\).
We define the shuffle product $*$ as

$$(\omega_1 \otimes \ldots \otimes \omega_p) * (\omega_{p+1} \otimes \ldots \otimes \omega_{p+q}) = \sum_{(p,q) \text{ shuffles } \sigma} (-1)^{A_{\sigma}} (\omega_{\sigma(1)} \otimes \omega_{\sigma(2)} \otimes \ldots \otimes \omega_{\sigma(p+q)})$$

where $A_{\sigma} = A_{\sigma}(\omega_1, \ldots, \omega_{p+q}) = \sum_{i<j \text{ and } \sigma(i) > \sigma(j)} (k_{\sigma(i)} - 1)(k_{\sigma(j)} - 1)$.

The shuffle product

$$* : B_{xy}^* \otimes B_{xy}^* \to B_{xy}^*$$

is a chain map. It is associative and $a * b = (-1)^{deg_a deg_b} b * a$.

Let $x = x_1 = x_2$. The inclusion $x \hookrightarrow X$ induces a chain map $e_x : B_{xx}(\Omega^*(X)) \to B_{xx}(\Omega^*(x))$.

Let $C$ be a complex equal $C$ in degree 0 and 0 in degree $n \neq 0$. The inclusion of $C$ in $\Omega^0(X)$ induces

$$p_{x_2 x_1} : B_{x_2 x_1}^* (C) \to B_{x_2 x_1}^* (\Omega^*(X)).$$

**Proposition 2.3.1.** (see also [W1]). $B_{x_2 x_1}^* (\Omega^*(X))$ equipped with the shuffle product $*$ is a differential $C$-algebra. The maps $M$, $t$, $e$ and $p$ are homomorphisms of differential $C$-algebras. We have

$$(M_{x_4 x_3 x_2} \otimes id_{B_{x_2 x_1}^*}) \circ M_{x_4 x_2 x_1} = (id_{B_{x_4 x_3}^*} \otimes M_{x_3 x_2 x_1}) \circ M_{x_4 x_3 x_1},$$

$$* \circ (p_{x_2 x_1} \circ e_{x_2}) \otimes id_{B_{x_2 x_1}^*} \circ M_{x_2 x_2 x_1} = id_{B_{x_2 x_1}^*},$$

$$* \circ id_{B_{x_2 x_1}^*} \otimes (p_{x_1 x_1} \circ e_{x_1}) \circ M_{x_2 x_1 x_1} = id_{B_{x_2 x_1}^*},$$

$$* \circ (t_{xx} \otimes id_{B_{x}^*}) \circ M_{xx} = p_{xx} \circ e_{x},$$

$$* \circ (id_{B_{xx}^*} \otimes t_{xx}) \circ M_{xx} = p_{xx} \circ e_{x}.$$
Corollary 2.3.2. (see also [W1]). Spec $H^0(B^\bullet_{xx})$ is an affine pro-algebraic pro-unipotent group scheme over $C$. Spec $H^0(B^\bullet_{yx})$ is a (right) Spec $H^0(B^\bullet_{xx})$-torsor and a (left) Spec $H^0(B^\bullet_{yy})$-torsor.

Proof. It follows immediately from Proposition 2.3.1 that Spec $H^0(B^\bullet_{xx})$ is an affine group scheme over $C$ and that Spec $H^0(B^\bullet_{yx})$ is a (right) Spec $H^0(B^\bullet_{xx})$ - torsor and a (left) Spec $H^0(B^\bullet_{yy})$ - torsor.

2.4. We define a map

$$\varepsilon : T_2 = \text{Tot}(\oplus_{n=0}^{\infty} \Omega^*(X)^{\otimes n}) \to B^\bullet(\Omega^*(X))$$

in such a way that $\varepsilon$ on $\Omega^{k_1}(X) \otimes \ldots \otimes \Omega^{k_n}(X)$ is a multiplication by $(-1)^{c(n)+k_1+k_3+k_5+\ldots}$. The map $\varepsilon$ is an isomorphism of chain complexes. We define the shuffle multiplication $\ast$, the coproduct $\mathcal{M}$ and the inverse $\iota$ on $T_2$ (and on $T_1$) setting :

$$\mathcal{M} \text{ on } T_2 : (\varepsilon \otimes \varepsilon)^{-1} \circ (\mathcal{M} \text{ on } B^\bullet) \circ \varepsilon$$

and similarly for $\iota$ and $\ast$. The maps $e_x$ and $p_{yx}$ can be defined in the same way.

One can also proceed in the following way. The projection on the constant cosimplicial space $p^\bullet : p^\bullet^{-1}(y,x) \to (y,x)$ induces

$$p_{yx} : \text{Tot}(\oplus_{n=0}^{\infty} \Omega^*(y,x)) \to \text{Tot}(\oplus_{n=0}^{\infty} \Omega^*(y \times X^n \times x)).$$

Let $(x)$ be a constant cosimplicial space equal one point $x$ in each degree. The inclusion

$$(x) \hookrightarrow p^\bullet^{-1}(x,x), \ x \to (x,x,\ldots,x) \in x \times X^n \times x$$

induces

$$\text{Tot}(\oplus_{n=0}^{\infty} \Omega^*(x \times X^n \times x)) \to \text{Tot}(\oplus_{n=0}^{\infty} \Omega^*(x)).$$

2.5. Let $G$ be a Lie group and let $g$ be a Lie algebra of $G$. Let $\pi : P \to X$ be a principal $G$-bundle equipped with the integrable connection given by a one form $\omega$ on $P$ with values in $g$. Let us set

$$\mathcal{P} := \pi_* \mathcal{O}_P.$$
Then \( \mathcal{P} \) is a vector bundle on \( X \) (an inductive limit of finite dimensional vector bundles) equipped with an integrable connection \( \nabla_\mathcal{P} \) induced by the connection on \( P \).

Let \( \Omega^\ast \) be the De Rham complex of sheaves of smooth complex valued differential forms on \( X \), let \( \Omega^\ast(\mathcal{P}) \) be the twisted De Rham complex and let \( \Omega^\ast(\mathcal{P})(X) \) be the twisted de Rham complex of global sections.

For each point \( x \in X \) we define an augmentation

\[
\eta_x : \Omega^0(\mathcal{P})(X) \to \mathcal{O}[P_x],
\]

where \( \eta_x(f) \) is a restriction of \( f \in \Omega^0(\mathcal{P})(X) \) to a fiber of \( \mathcal{P} \) over \( x \).

Let \( y \in X \). Let us choose \( y' \in \pi^{-1}(y) \). We define an augmentation

\[
\varepsilon_{y'} : \Omega^0(\mathcal{P})(X) \to C
\]
setting \( \varepsilon_{y'}(f) := f(y') \).

Let \( (y, x) \in X \times X \). Let

\[
\partial : C \otimes (\Omega^\ast(\mathcal{P})(X))^\otimes n \otimes \mathcal{O}[P_x] \to C \otimes (\Omega^\ast(\mathcal{P})(X))^\otimes n \otimes \mathcal{O}[P_x]
\]
be a differential of a tensor product of twisted De Rham complexes.

We define

\[
\delta_i : C \otimes (\Omega^\ast(\mathcal{P})(X))^\otimes n \otimes \mathcal{O}[P_x] \to C \otimes (\Omega^\ast(\mathcal{P})(X))^\otimes (n-1) \otimes \mathcal{O}[P_x]
\]
in the following way:

\[
\delta_i(f_{n+1} \otimes \omega_n \otimes \ldots \otimes \omega_1 \otimes f_0) = f_{n+1} \otimes \omega_n \otimes \ldots \otimes \omega_{i+1} \wedge \omega_i \otimes \ldots \otimes \omega_1 \otimes f_0
\]
for \( i \neq 0, n \);

\[
\delta_0(f_{n+1} \otimes \omega_n \otimes \ldots \otimes \omega_1 \otimes f_0) = f_{n+1} \otimes \omega_n \otimes \ldots \otimes \omega_2 \otimes \eta_x(\omega_1)f_0;
\]

\[
\delta_n(f_{n+1} \otimes \omega_n \otimes \ldots \otimes \omega_1 \otimes f_0) = f_{n+1}\varepsilon_{y'}(\omega_n) \otimes \omega_{n-1} \ldots \otimes \omega_1 \otimes f_0.
\]
(\wedge is induced by the exterior product and the multiplication in \mathcal{P}.)

Let us set

$$
\delta := \sum_{i=0}^{n} (-1)^i \delta_i : C \otimes (\Omega^*(\mathcal{P})(X))^{\otimes n} \otimes \mathcal{O}[P_x] \rightarrow C \otimes (\Omega^*(\mathcal{P})(X))^{\otimes (n-1)} \otimes \mathcal{O}[P_x].
$$

We have \( \partial^2 = \delta^2 = \partial \delta - \delta \partial = 0 \). Therefore \( \otimes_{n=0}^{\infty} C \otimes (\Omega^*(\mathcal{P})(X))^{\otimes n} \otimes \mathcal{O}[P_x] \) is a double complex and let \( T_{y,x} \) be its total complex.

### 2.6. We define a bar construction

$$
B^\bullet_{y,x}(C; \Omega^*(\mathcal{P})(X) ; \mathcal{O}[G])
$$

being the same bigraded module as \( T_{y,x} \) with the differential \( d' \) defined as in 2.2. We shall denote \( B^\bullet_{y,x}(C; \Omega^*(\mathcal{P})(X) ; \mathcal{O}[G]) \) shortly by \( B^\bullet_{y,x} \).

The map \( \varepsilon : T_{y,x} \rightarrow B^\bullet_{y,x}(C; \Omega^*(\mathcal{P})(X) ; \mathcal{O}[G]) \) given by a multiplication by \( (-1)^{c(n)+k_1+k_3+k_5+\ldots} \) on \( (\Omega^{k_1}(X) \otimes \mathcal{O}[G]) \otimes \ldots \otimes (\Omega^{k_n}(X) \otimes \mathcal{O}[G]) \otimes \mathcal{O}[G] \) is an isomorphism of chain complexes.

The choice of \( x' \in \pi^{-1}(x) \) identifies \( P_x \) - the fiber of \( P \) over \( x \) - with \( G \), hence the fiber of \( \mathcal{P} \) over \( x \) is identified with \( \mathcal{O}[G] \).

We assume that the principal \( G \) - bundle \( \pi : P \rightarrow X \) is trivial. Then \( \Omega^*(\mathcal{P})(X) = \Omega^*(X) \otimes \mathcal{O}[G] \).

The action of \( G \) on \( P \) induces \( \mathcal{P} \rightarrow \mathcal{P} \otimes \mathcal{O}[G] \), \( f(p) \rightarrow f(pg) \). Let \( x \in X \) be such that the fiber of \( \mathcal{P} \) over \( x \) is \( \mathcal{O}[G] \).

We define

$$
\mathcal{M}_{y,x} : B^\bullet_{y,x} \rightarrow B^\bullet_{y,x} \otimes B_{x,x}
$$

by the formula

$$
\mathcal{M}_{y,x}(f_0 \otimes \omega_1(p_1) \otimes \ldots \otimes \omega_1(p_1) \otimes f_{n+1}(g_{n+1})) =
\sum_{i=0}^{n} (f_0 \otimes \omega_1(p_1) \otimes \ldots \otimes \omega_i(p_i) \otimes 1) \otimes \\
(1 \otimes \omega_{i+1}(p'_{i+1}g_{i+1}) \otimes \ldots \otimes \omega_n(p'_n g_{i+1}) \otimes f_{n+1}(g'_{n+1} g_{i+1})) .
$$

We define

$$
\iota : B^\bullet_{x,x} \rightarrow B^\bullet_{x,x}
$$

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by the formula
\[
\iota(f_0 \otimes \omega_1(p_1) \otimes \ldots \otimes \omega_1(p_1) \otimes f_{n+1}(g_{n+1})) =
\]
\[
(-1)^{\frac{n(n+1)}{2}+(n+1)(k_1+\ldots+k_n)+\sum_{i<j} k_i k_j} f_0 \otimes \omega_n(p_n g_{n+1}^{-1}) \otimes \ldots \otimes \omega_1(p_1 g_{n+1}^{-1}) \otimes f_{n+1}(g_{n+1}^{-1}).
\]

We define the shuffle product
\[
*: B_{yx}^* \otimes B_{yx}^* \to B_{yx}^*
\]
by the formula
\[
(f_0 \otimes \omega_1 \otimes \ldots \otimes \omega_p \otimes f_{p+1}) \ast (g_p \otimes \omega_{p+1} \otimes \ldots \otimes \omega_{p+q} \otimes g_{p+q+1}) =
\]
\[
\sum_{(p,q)\text{-shuffles } \sigma} (-1)^{A_{\sigma}} f_0 g_p \otimes \omega_\sigma(1) \otimes \ldots \otimes \omega_\sigma(p+q) \otimes f_{p+1} g_{p+q+1}.
\]

The sign \((-1)^{A_{\sigma}}\) is defined as in 2.3.

The inclusion \(x \hookrightarrow X\) induces
\[
e_x : B_{x,x}^* (C; \Omega^*(\mathcal{P})(X); \mathcal{O}[G]) \to B_{x,x}^* (C; \mathcal{O}[G]; \mathcal{O}[G]).
\]

\(B_{y,x}^\bullet\) equipped with the shuffle product is a \(C\)-algebra, hence we have a morphism of \(C\)-algebras
\[
p_{yx} : C \to B_{y,x}^*.
\]

**Proposition 2.6.1.** \(B_{y,x}^\bullet (C; \Omega^*(\mathcal{P})(X); \mathcal{O}[G])\) equipped with the shuffle product \(*\) is a differential \(C\)-algebra. The maps \(M, \iota, e\) and \(p\) are homomorphims of differential \(C\)-algebras. We have

\[
(M_{yxx} \otimes id_{B_{yx}^*}) \circ M_{yxx} = (id_{B_{yx}^*} \otimes M_{xxy}) \circ M_{yxx},
\]

\[
* \circ (p_{xx} \circ e_x) \otimes id_{B_{yx}^*} \circ M_{xxy} = id_{B_{yx}^*},
\]

\[
* \circ id_{B_{yx}^*} \otimes (p_{xx} \circ e_x) \circ M_{xxy} = id_{B_{yx}^*},
\]

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\* \circ (t_{xx} \otimes \text{id}_{B^*_x}) \circ \mathcal{M}_{xx} = p_{xx} \circ e_x, \\
\* \circ (\text{id}_{B^*_x} \otimes t_{xx}) \circ \mathcal{M}_{xx} = p_{xx} \circ e_x.

**Proof.** One checks the formulas by calculations.

Passing to cohomology the maps $\mathcal{M}$, $\iota$ and $e$ induce

$$
\mathcal{M}_{yx} : H^0(T_{yx}) \rightarrow H^0(T_{yx}) \otimes H^0(T_{xx}),
$$

$$
\iota_{yx} : H^0(T_{yx}) \rightarrow H^0(T_{xy}) \text{ and } e_x : H^0(T_{xx}) \rightarrow C.
$$

$H^0(T_{yx})$ equipped with the shuffle product is a $C$-algebra. Hence we have a homomorphism of $C$-algebras

$$
p_{yx} : C \rightarrow H^0(T_{yx}).
$$

**Proposition 2.5.2.** $H^0(T_{yx})$ equipped with the shuffle product is a $C$-algebra. The maps $\mathcal{M}_{yx}$, $\iota_{yx}$ and $e_x$ are homomorphisms of differential $C$-algebras. We have

$$
(\mathcal{M}_{yx} \otimes \text{id}_{H^0(T_{xx})}) \circ \mathcal{M}_{yx} = (\text{id}_{H^0(T_{yx})} \otimes \mathcal{M}_{xx}) \circ \mathcal{M}_{yx},
$$

$$
\* \circ (p_{xx} \circ e_x) \otimes \text{id}_{H^0(T_{xx})} \circ \mathcal{M}_{xx} = \text{id}_{H^0(T_{xx})},
$$

$$
\* \circ \text{id}_{H^0(T_{yx})} \otimes (p_{xx} \circ e_x) \circ \mathcal{M}_{yx} = \text{id}_{H^0(T_{xx})},
$$

$$
\* \circ (t_{xx} \otimes \text{id}_{H^0(T_{xx})}) \circ \mathcal{M}_{xx} = p_{xx} \circ e_x, \\
\* \circ (\text{id}_{H^0(T_{xx})} \otimes t_{xx}) \circ \mathcal{M}_{xx} = p_{xx} \circ e_x.
$$

_____________

2.6. We define a bar construction

$$
B_{y,x}^*(\mathcal{O}[P_y] \otimes \Omega^*(X) \otimes \mathcal{O}[G]; \mathcal{O}[G]) \text{ being the same bigraded module as } T_{y,x} \text{ with the differential } d' \text{ defined as in 2.2. The map } \varepsilon : T_{y,x} \rightarrow B_{y,x}^*(\Omega^*(X) \otimes \mathcal{O}[G]; \mathcal{O}[G]) \text{ given by a}
$$
multiplication by \((-1)^{c(n)+k_1+k_3+k_5+\ldots} \) on \(\Omega^k(X) \otimes \mathcal{O}[G] \otimes \ldots \otimes (\Omega^n(X) \otimes \mathcal{O}[G]) \otimes \mathcal{O}[G]\)
is an isomorphism of chain complexes.

The comultiplication \(\mathcal{M}\), the inverse \(\iota\) and the product \(*\) are given on \(B^*(\Omega^*(X) \otimes \mathcal{O}[G]; \mathcal{O}[G])\) with the sign convention as in 2.2.

2.7. We have the following analogues of 2.3.1 and 2.3.2.

**Proposition 2.7.1.** \(T_{x_1,x_2}(\Omega^*(X))\) equipped with the shuffle product \(*\) is a differential \(C\)-algebra. The maps \(\mathcal{M}\), \(\iota\), \(e\) and \(p\) are homomorphisms of differential \(C\)-algebras. We have

\[
(M_{x_1,x_2,x_3} \otimes id_{T_{x_1,x_2}}) \circ M_{x_1,x_2,x_3} = (id_{T_{x_1,x_2}} \otimes M_{x_1,x_2,x_3}) \circ M_{x_1,x_2,x_3},
\]

\[
* \circ (p_{x_1,x_2} \circ e_{x_2}) \otimes id_{T_{x_1,x_2}} \circ M_{x_1,x_2,x_3} = id_{T_{x_1,x_2}},
\]

\[
* \circ id_{T_{x_1,x_2}} \otimes (p_{x_1,x_2} \circ e_{x_1}) \circ M_{x_1,x_2,x_3} = id_{T_{x_1,x_2}},
\]

\[
* \circ (\iota_{x_1,x_2} \otimes id_{T_{x_1,x_2}}) \circ M_{x_1,x_2} = p_{x_1,x_2} \circ e_{x_1},
\]

\[
* \circ (id_{T_{x_1,x_2}} \otimes \iota_{x_1,x_2}) \circ M_{x_1,x_2} = p_{x_1,x_2} \circ e_{x_1}.
\]

Corollary 2.5.3. We have

i). \(\text{Spec} \, H^0(T_{x_1,x_2})\) is an affine pro-algebraic group scheme over \(C\).

ii). \(\text{Spec} \, H^0(T_{y_1,y_2})\) is a (right) \(\text{Spec} \, H^0(T_{x_1,x_2})\)-torsor.

iii). The group scheme \(\text{Spec} \, H^0(T_{x,x})\) is an extension of \(\text{Spec} \, \mathcal{O}[G]\) by an affine pro-unipotent group over \(C\).

*Proof.* It follows from Proposition 2.5.2 that \(\text{Spec} \, H^0(T_{x_1,x_2})\) is a group scheme over \(C\) and that \(\text{Spec} \, H^0(T_{y_1,y_2})\) is a \(\text{Spec} \, H^0(T_{x_1,x_2})\)-torsor.
First we show that there is a surjective morphism of group schemes

$$\text{Spec } H^0(T_{xx}) \to \text{Spec } \mathcal{O}[G].$$

Observe that $T_{xx}^{00} = \mathcal{O}[G] \otimes \mathcal{O}[G]$, $T_{xx}^{10} = \mathcal{O}[G] \otimes \mathcal{O}[G]$ and $T_{xx}^{01} = \mathcal{O}[G] \otimes \mathcal{O}_X(\mathcal{P})(X) \otimes \mathcal{O}[G]$. The differential

$$\delta : T_{xx}^{01} \to T_{xx}^{00}$$

is given by $\delta(f \otimes g \otimes h) = (f \eta_x(g) \otimes h - f \otimes \eta_x(g)h)$, where $f, h \in \mathcal{O}[G]$ and $g \in \mathcal{O}_X(\mathcal{P})$. Therefore $\mathcal{O}[G]$ is a Hopf subalgebra of $H^0(T_{xx})$ and we have an exact sequence

$$1 \to U \to \text{Spec } H^0(T_{xx}) \xrightarrow{\varphi_n} \text{Spec } \mathcal{O}[G] \to 1,$$

where $U := \ker \varphi$.

Let us set

$$G^{(n)} := \text{Spec } F_n H^0(T_{xx}).$$

The inclusion

$$F_{n-1} H^0(T_{xx}) \to F_n H^0(T_{xx})$$

induces a surjective morphism of group scheme

$$\varphi_n : G^{(n)} \to G^{(n-1)}.$$

Observe that $G^{(0)} = G$ and $\text{Spec } H^0(T_{xx}) = \lim_{n \to \infty} G(n)$. Let us set

$$U^{(n)} := \ker(G^{(n)} \to G^{(0)}).$$
We have $U = \varprojlim U^n$. We shall show that for any $n$, the group $A_n := \ker(\varphi_n : U^n \to U^{n-1})$ is abelian. This implies that $U$ is pro-unipotent.

Observe that $A_1 = U^{(1)}$. We have an exact sequence of groups

$$1 \to U^{(1)} \to G^{(1)} \to G^{(0)} \to 1$$

and the corresponding sequence of coordinate rings

$$\mathcal{O}[G] \hookrightarrow F_1 H^0(T_{xx}) \to F_1 H^0(T_{xx})/I(U^{(1)}).$$

The kernel of the augmentation homomorphism $\varepsilon : \mathcal{O}[G] \to C \ (f \to f(e))$ is contained in $I(U^{(1)})$. We have $\mathcal{M}(1 \otimes w \otimes 1) = (1 \otimes w \otimes 1) \otimes (1 \otimes 1) + \sum_i (1 \otimes f_i) \otimes (1 \otimes w^i \otimes 1)$, where $m : \Omega^1_X(P)(X) \to \mathcal{O}[G] \otimes \Omega^1_X(P)(X)$ is induced by the action of $G$ on $P$ and $m(w) = \sum f_i \otimes w_i$. Observe that $1 \otimes f_i \equiv 1 \otimes f_i(e) \mod I(U^{(1)})$. Hence $\mathcal{M}(1 \otimes w \otimes 1) = (1 \otimes w \otimes 1) \otimes (1 \otimes 1) + (1 \otimes 1) \otimes (1 \otimes w \otimes 1) \mod I(U^{(1)})$. This implies that $U^{(1)}$ is abelian.

To show that $A_n$ is abelian one observes that $A_n := \ker(G^{(n)} \to G^{(n-1)})$ and one repeats the proof given above for $A_1$. 

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§3. Universal variation of mixed Hodge structures.

3.0. Let $X$ be a topological space. The inclusion of simplicial sets

$$\partial \Delta[1] \hookrightarrow \Delta[1]$$

induces a morphism of cosimplicial spaces

$$p^\bullet : X^{\Delta[1]} \to X^{\partial \Delta[1]}.$$ 

Observe that $X^{\partial \Delta[1]}$ is a constant cosimplicial space equal $X \times X$ in each degree. Let $x$ and $y$ be two points of $X$. Then constant cosimplicial spaces $(X \times x)^\bullet$, $(y \times X)^\bullet$ and $(y \times x)^\bullet$ are cosimplicial subspaces of $X^{\partial \Delta[1]}$. Let us set

$$(X; y)^{(\Delta[1], 1)} := p^{\bullet-1}((y \times X)^\bullet), \quad (X; x)^{(\Delta[1]; 0)} := p^{\bullet-1}((X \times x)^\bullet),$$

$$(X; x, y)^{(\Delta[1]; 0, 1)} := p^{\bullet-1}((y \times x)^\bullet) \quad \text{and} \quad (X; x)^{(S^1; 1)} := p^{\bullet-1}((x \times x)^\bullet).$$

We shall use also the notation $(y \times X)^\bullet = (X; y)^{(\partial \Delta[1], 1)}$. We recall that the cosimplicial space $X^{\Delta[1]}$ is given by

$$X \times X^0 \xrightarrow{\delta^1} X \times X^0 \times X \xrightarrow{\delta^2} X \times X^2 \times X \ldots,$$

where

$$\delta^i : X \times X^n \times X \to X \times X^{n+1} \times X, \quad \delta^i(x_{n+1}, x_n, \ldots, x_1, x_0) = (x_{n+1}, x_n, \ldots, x_i, x_i, \ldots, x_1, x_0).$$

3.1. We shall consider parallelly two cases:

i) $X$ is a smooth holomorphic variety and $\Omega_X^*$ is a holomorphic De Rham complex. $G$ is an affine complex reductive algebraic group and $\pi : P \to X$ is a holomorphic principal $G$-bundle equipped with a holomorphic integrable connection.

ii) $X$ is a smooth quasi-projective scheme of finite type over a field $k$ of characteristic zero, $G$ is an affine reductive algebraic group over $\text{Spec} k$ and $\pi : P \to X$ is a principal $G$-bundle locally trivial in Zariski topology (resp. etale topology, resp. flat topology) of $X$. $\Omega_X^*$ is
the algebraic De Rham complex on $X_{\text{Zar}}$ (resp. $X_{\text{et}}$, resp. $X_{\text{flat}}$) - $X$ equipped with the corresponding topology.

Let $\mathcal{P} \coloneqq \pi_* \mathcal{O}_P$. This is an inductive limit of finite dimensional vector bundles on $X$ in a corresponding topology. The sheaf $\mathcal{P}$ is a sheaf of $\mathcal{O}_X$-algebras. The connection on the principal $G$-bundle $\pi : P \rightarrow X$ induces a multiplicative integrable connection $\nabla$ on $\mathcal{P}$ i.e.

$$\nabla(a \cdot b) = \nabla(a) \cdot b + a \cdot \nabla(b),$$

where $a$ and $b$ are sections of $\mathcal{P}$. Hence the tensor products of sheaves $\mathcal{P}^{\otimes (n+2)}$ on $X \times X^n \times X$ define a sheaf $\mathcal{P}^{\otimes (n+2)}$ on a cosimplicial space $X^{\Delta[1]}$. The twisted De Rham complexes $\Omega^n_{X_{\text{Zar}}}((\mathcal{P}^{\otimes (n+2)}))$ define a twisted De Rham complex $\Omega^n_{X_{\Delta[1]}}((\mathcal{P}^{\otimes (n+2)}))$ on $X^{\Delta[1]}$. The relative twisted De Rham complexes $\Omega^n_{(X \times X^{\Delta[1]})/X}(\mathcal{P}^{\otimes (n+2)})$ define a relative twisted De Rham complex

$$\Omega^* := \Omega^n_{X^{\Delta[1]}/(X \times X)}((\mathcal{P}^{\otimes (n+2)}))$$

on a cosimplicial space $X^{\Delta[1]}$.

We shall study the cohomology sheaves $H^i(tR(p^*)_* \Omega^*)$ of $\mathcal{O}_{X \times X}$-modules, where $p^* : X^{\Delta[1]} \rightarrow X^{\partial \Delta[1]}$.

The fiber of $H^i(tR(p^*)_* \Omega^*)$ over $(y, x) \in X \times X$ can be describe in the following way.

The restriction of $\Omega^n_{X_{\text{Zar}}}((\mathcal{P}^{\otimes (n+2)}))$ to $y \times X^n \times x$ is $\mathcal{O}[P_y] \otimes \Omega_{y \times X^n \times x}(\mathcal{P}^{\otimes n}) \otimes \mathcal{O}[P_x]$. Hence the fiber of $H^i(tR(p^*)_* \Omega^*)$ over $(y, x)$ is $H^i(tR\Gamma_{(X,y,x)}^{(\Delta[1];y,x)}(\mathcal{O}[P_y] \otimes \mathcal{P}^{\otimes *} \otimes \mathcal{O}[P_x]))$.

3.2. We assume that $\mathcal{O}[G]$ carries a Hodge structure of weight 0 and that the structure morphisms and the multiplication are morphisms of Hopf structures. We assume that $\mathcal{P}$ equipped with the connection deduced from the connection on $\pi : P \rightarrow X$ carries a variation of Hodge structures of weight 0 compatible with multiplication in $\mathcal{P}$ and with the action of $\mathcal{O}[G]$ on $\mathcal{P}$.

3.2.1. Our aim is to show that the cohomology sheaves $H^i(tR(p^*)_* \Omega^*)$ carry a variation of mixed Hodge structures over $X$. Next we shall show that $H^0(tR(p^*)_* \Omega^*)$ is the sheaf of $\mathcal{O}_{X \times X}$-algebras and that the variation of mixed Hodge structures is compatible with multiplication. Let $\mathcal{H}$ be a fiber of $H^0(tR(p^*)_* \Omega^*)$ over $(x, x)$. We shall show that $\mathcal{H}$ is
a Hopf algebra and that the Hopf algebra structure and the action are compatible with mixed Hodge structures.

3.3. We assume that $X$ is a smooth projective variety over $C$.

Let $V$ be a variation of Hodge structures of weight $k$ over $X$ and let $\{ F^p V \}_p$ be a Hodge filtration of $V$. Let us equip the complex $\Omega^*_X(V)$ with a filtration $\{ F^p \}_p$ in the following way:

$$ F^p(\Omega^*_X(V)) := \Omega^*_X(F^{p-r}(V)). $$

Let $\mathcal{V}_0$ be a rational lattice in $\mathcal{V}$. It is a local system of vector spaces over $Q$. We have

$$ \ker(\nabla : V \to \Omega^1_X(V)) \approx \mathcal{V}_0 \otimes C. $$

Hence there is a quasi-isomorphism $\alpha : \mathcal{V}_0 \otimes C \to \Omega^*_X(V)$. It follows from [Z] Theorem 2.9 that

$$(\mathcal{V}_0, (\Omega^*_X(V), F), \alpha : \mathcal{V}_0 \otimes C \to \Omega^*_X(V))$$

is a cohomological Hodge complex of weight $k$ over $X$.

3.3.1. For each $n$, $P^{\otimes n}$ is a variation of Hodge structures of weight zero on $y \times X^n \times x$.

Hence

$$ (\mathcal{O}[P_y]_0 \otimes P^{\otimes n}_0 \otimes \mathcal{O}[P_x]_0, (\Omega^*_y \times X^n \times x (\mathcal{O}[P_y] \otimes P^{\otimes n} \otimes \mathcal{O}[P_x]), F), $$

$$ \alpha_n : \mathcal{O}[P_y]_0 \otimes P^{\otimes n}_0 \otimes \mathcal{O}[P_x]_0 \otimes C \to \Omega^*_y \times X^n \times x (\mathcal{O}[P_y] \otimes P^{\otimes n} \otimes \mathcal{O}[P_x])) $$

is a cohomological Hodge complex of weight zero on $y \times X^n \times x$.

3.3.2. The complexes $\Omega^*_y \times X^n \times x (\mathcal{O}[P_y] \otimes P^{\otimes n} \otimes \mathcal{O}[P_x])$ equipped with the filtration $F$ form a complex $\Omega^*_z (X; x, y)^{(\Delta[1]; 0, 1)} (\mathcal{O}[P_y] \otimes P^{\bullet} \otimes \mathcal{O}[P_x])$ equipped with the filtration $F$ on the cosimplicial space $(X; x, y)^{(\Delta[1]; 0, 1)}$. The sheaves $\mathcal{O}[P_y]_0 \otimes P^{\otimes n}_0 \otimes \mathcal{O}[P_x]_0$ form a sheaf $\mathcal{O}[P_y]_0 \otimes P^{\otimes^0 \bullet} \otimes \mathcal{O}[P_x]_0$ on $(X; x, y)^{(\Delta[1]; 0, 1)}$. The morphisms $\alpha_n$ induce a quasi-isomorphism

$$ \alpha : \mathcal{O}[P_y]_0 \otimes P^{\otimes^0 \bullet} \otimes \mathcal{O}[P_x]_0 \otimes C^{\otimes^0_1} \Omega^*_z (X; x, y)^{(\Delta[1]; 0, 1)} (\mathcal{O}[P_y]_0 \otimes P^{\otimes^0 \bullet} \otimes \mathcal{O}[P_x]_0). $$

Hence the triple

$$(\mathcal{O}[P_y]_0 \otimes P^{\otimes^0 \bullet} \otimes \mathcal{O}[P_x]_0, (\Omega^*_z (X; x, y)^{(\Delta[1]; 0, 1)} (\mathcal{O}[P_y] \otimes P^{\otimes^0 \bullet} \otimes \mathcal{O}[P_x]), F),$$

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\[ \alpha : \mathcal{O}[P_y]_0 \otimes \mathcal{P}^\bullet_0 \otimes \mathcal{O}[P_x]_0 \otimes C \rightarrow \Omega^*_{(X;x,y)(\Delta[1];0,1)}(\mathcal{O}[P_y] \otimes \mathcal{P}^\bullet \otimes \mathcal{O}[P_x]) \]

is a cohomological Hodge complex on a cosimplicial space \((X;x,y)(\Delta[1];0,1)\).

3.3.3. Let \(K = (K^{i,j})\) be a simplicial complex \((j\) is a simplicial degree). We define a filtration \(\partial(W,R)\) on the total complex \(tK\) in the following way:

\[ \partial(W,R)_n(tK) := \oplus_{j<n+1} K^{i,j}. \]

Applying the functor \(R\Gamma^*\) (\(R\Gamma\) functorially in each degree) we get the following result.

**Theorem 3.3.4.** The triple

\[ ((tR\Gamma^*(\mathcal{O}[P_y]_0 \otimes \mathcal{P}^\bullet_0 \otimes \mathcal{O}[P_x]_0); \partial(W,R)), (tR\Gamma^*\Omega^*_{(X;x,y)(\Delta[1];0,1)}(\mathcal{O}[P_y] \otimes \mathcal{P}^\bullet \otimes \mathcal{O}[P_x]), \partial(W,R), F), \]

\[ tR\Gamma^*\alpha := (tR\Gamma^*(\mathcal{O}[P_y]_0 \otimes \mathcal{P}^\bullet_0 \otimes \mathcal{O}[P_x]_0) \otimes C, \partial(W,R)) \rightarrow \]

\[ tR\Gamma^*\Omega^*_{(X;x,y)(\Delta[1];0,1)}(\mathcal{O}[P_y] \otimes \mathcal{P}^\bullet \otimes \mathcal{O}[P_x]), \partial(W,R)) \]

is a mixed Hodge complex.

**Corollary 3.3.5.** The cohomology groups \(H^i(tR\Gamma^*\Omega^*_{(X;x,y)(\Delta[1];0,1)}(\mathcal{O}[P_y] \otimes \mathcal{P}^\bullet \otimes \mathcal{O}[P_x]))\) are equipped with mixed Hodge structures.

3.4. We give a relative version of the construction from 3.3.

The Hodge filtration on \(\mathcal{P}\) induces the Hodge filtration \(\{F^p\}\) on \(\Omega^*\). Observe also that the complex \(\Omega^*\) is a resolution of \((p^{*-1})\mathcal{O}_{X \times X} \otimes Q \mathcal{P}_0^{\otimes s+2}\).

**Proposition 3.4.1.** The triple

\[ (\mathcal{P}_0^{\otimes s+2}; (\Omega^*, F); \alpha^* : (p^{*-1})\mathcal{O}_{X \times X} \otimes Q \mathcal{P}_0^{\otimes s+2} \rightarrow \Omega^*) \]

is a relative w.r.t. \(p^* : X^{\Delta[1]} \rightarrow X^{0\Delta[1]}\) cohomological Hodge complex on \(X^{\Delta[1]}\).

**Proposition 3.4.2.** The triple

\[ ((tRp^*\mathcal{P}_0^{\otimes s+2}, \partial(W,R)), (tRp^*\Omega^*, \partial(W,R), F), \alpha := tRp^*\alpha^* : tRp^*\mathcal{P}_0^{\otimes s+2} \otimes Q\mathcal{O}_{X \times X} \rightarrow tRp^*\Omega^*) \]

is a mixed Hodge complex on \(X \times X\).

We denote by \(W\) the filtration induced by \(\partial(W,R)\) on cohomology groups.
Theorem 3.4.3. The triple

\[ ((H^i(tR_p^*\mathcal{P}_0^\otimes (\bullet+2)); W), (H^i(tR_p^*\Omega^*), W, F), \]
\[ \alpha : H^i(tR_p^*\mathcal{P}_0^\otimes (\bullet+2)) \otimes_Q \mathcal{O}_{X \times X}, W) \rightarrow (H^i(tR_p^*\Omega^*), W)) \]

is a variation of mixed Hodge structures over \( X \times X \).

Proof. One need only to construct an integrable connection
\[ \nabla : H^i(tR_p^*\Omega^*) \rightarrow \Omega^1_{X \times X} \otimes_{\mathcal{O}_{X \times X}} H^i(tR_p^*\Omega^*) \]
which has the required properties. This is done in the next sections.

3.6. The Gauss-Manin connection.

Let us set:

\[ G_{n}^i(\Omega^*_{X+n+2}(\mathcal{P}^\otimes (n+2))) := \]
\[ \text{image}(\Omega^*_{X+n+2}(\mathcal{P}^\otimes (n+2)) \otimes_{\mathcal{O}_{X \times X}} (p^n)^*\Omega^i_{X \times X} \rightarrow \Omega^*_{X+n+2}(\mathcal{P}^\otimes (n+2))). \]

The filtrations \{G_{n}^i\} of \( \Omega^*_{X+n+2}(\mathcal{P}^\otimes (n+2)) \) define a filtration \{G_{1}^i\} of \( \Omega^*_{X+\Delta[1]}(\mathcal{P}^\otimes (\bullet+2)) \). We have
\[ gr_{i}^i(\Omega^*_{X+\Delta[1]}(\mathcal{P}^\otimes (\bullet+2))) = \Omega^{* -i} \otimes_{\mathcal{O}_{X \times \Delta[1]}} (p^n)^*\Omega^i_{(X \times X)\bullet}. \]

Consider the functor \( t \circ R(p^*) \) from the category of complexes of abelian sheaves on \( X^{\Delta[1]} \) to the category of complexes of abelian sheaves on \( X \times X \). We apply the spectral sequence of a filtered object to \( \Omega^*_{X+\Delta[1]}(\mathcal{P}^\otimes (\bullet+2)) \) and we get a spectral sequence converging to \( H^i(tR(p^*)_*\Omega^*_{X+\Delta[1]}(\mathcal{P}^\otimes (\bullet+2))) \) such that \( E^{p,q}_1 = \Omega^{p}_{X \times X} \otimes_{\mathcal{O}_{X \times X}} H^q(tR(p^*)_*\Omega^*) \). Let \( d^{p,q}_1 : E^{p,q}_1 \rightarrow E^{p+1,q}_1 \) be the differential on \( E_1 \) -terms.

Theorem 3.6.1. Let \( X \) be a smooth scheme of finite type over a field \( k \) of characteristic zero. We have:

i). The differential
\[ d^{0,q}_1 : H^q(tR_p^*\Omega^*) \rightarrow \Omega^1_{X \times X} \otimes_{\mathcal{O}_{X \times X}} H^q(tR_p^*\Omega^*) \]
is an integrable connection on \( H^q(tR_p^*\Omega^*) \). If \( f \) and \( g \) are sections of \( H^q(tR_p^*\Omega^*) \) and \( H^q(tR_p^*\Omega^*) \) respectively then
\[ d^{0,q+q'}_1 (f \cdot g) = d^{0,q}_1 (f) \cdot g + (-1)^q f \cdot d^{0,q'}_1 (g) \]

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ii). Let $F^i$ be the Hodge filtration of $H^q(tRp^\bullet \Omega^*)$. Then we have

$$d_1^{0,q}(F^i H^q(tRp^\bullet \Omega^*)) \subset \Omega^1_{X \times X} \otimes_{\mathcal{O}_{X \times X}} F^{i-1} H^q(tRp^\bullet \Omega^*).$$

iii). Assume that $X$ is also projective. Then we have

$$d_1^{0,q}(W_n H^q(tRp^\bullet \Omega^*)) \subset \Omega^1_{X \times X} \otimes_{\mathcal{O}_{X \times X}} W_n H^q(tRp^\bullet \Omega^*).$$

iv). The multiplication and the action of $\mathcal{H}$ on $H^0(tRp^\bullet \Omega^*) \rightarrow \mathcal{H} \otimes H^0(tRp^\bullet \Omega^*)$ are flat.

v). The sheaves $H^q(tRp^\bullet \Omega^*)$ are direct limits of vector bundles on $X \times X$.

vi). Let $K : k$ be an extension of fields. Then the cohomology sheaves and the connection $d_1^{0,q}$ for $X_K = X \times_{\text{Spec } k} \text{Spec } K$ are obtained from the ones for $X$ by applying the functor $- \otimes_k K$.

vii). Let $X$ be a smooth holomorphic variety and let $\Pi^{\otimes (\bullet + 2)} := \ker(\nabla : \mathcal{P}^{\otimes (\bullet + 2)} \rightarrow \Omega^*_{X \Delta[1]}(\mathcal{P}^{\otimes (\bullet + 2)}).$ Then $H^q(tRp^\bullet \Pi^{\otimes (\bullet + 2)}(X \Delta[1]))$ is the sheaf of flat sections of the connection $d_1^{0,q}$ on $H^q(tRp^\bullet \Omega^*)$.

**Proof.** The complex $tRp^\bullet \Omega^*_{X \Delta[1]}(\mathcal{P}^{\otimes (\bullet + 2)})$ is equipped with shuffle product. The filtration $\{G^i\}$ is compatible with the multiplicative structure. Hence the spectral sequence is multiplicative and

$$d_1^{0,q+q'}(f \cdot g) = d_1^{0,q}(f) \cdot g + (-1)^q f \cdot d_1^{0,q'}(g)$$

if $f \in H^q(tRp^\bullet \Omega^*)$ and $g \in H^{q'}(tRp^\bullet \Omega^*)$.

The differential $d_1$ has a bidegree $(1, 0)$ therefore for every $q$ we get a complex

$$\ldots \rightarrow \mathcal{O}_{x \times x} \otimes_{\mathcal{O}_{x \times x}} H^q(tRp^\bullet \Omega^*) \rightarrow \Omega^1_{x \times x} \otimes_{\mathcal{O}_{x \times x}} H^q(tRp^\bullet \Omega^*) \rightarrow \Omega^2_{x \times x} \otimes_{\mathcal{O}_{x \times x}} H^q(tRp^\bullet \Omega^*) \rightarrow \ldots.$$ 

Consider the diagram of cosimplicial spaces

$$\begin{array}{ccc}
X^{\Delta[1]} & \rightarrow & X^{\partial \Delta[1]} = (X \times X)^\bullet \\
\downarrow p^\bullet & & \downarrow \text{id} \\
X^{\partial \Delta[1]} & \rightarrow & X^{\partial \Delta[1]}
\end{array}$$
The horizontal map $p^\bullet$ induces a map of complexes of sheaves

$$\Omega_{X^{\partial[1]}}^\bullet \to (p^\bullet)_* \Omega_{X^{\partial[1]}}^\bullet \left( \mathcal{P}^{\otimes (\bullet+2)} \right)$$

given by

$$\Omega_{X \times X}(\mathcal{U}) \ni \omega \to (p^n)^* \omega \otimes 1 \in \Omega_{X^{n+2}}^\bullet \left( \mathcal{P}^{\otimes (n+2)} \right).$$

We repeat the construction of the spectral sequence for the vertical morphism $id$. The pair of horizontal morphisms $(p^\bullet, id)$ induces a map of corresponding spectral sequences. On $E^{1,0}_1$-terms we get a morphism of $E^{1,0}_1(id) = \Omega_{X \times X}$ into the complex $E^{1,0}_1(p^\bullet) = (*_0)$. Let $f \in \mathcal{O}_{X \times X}$ and $s \in H^q(tR_{p^\bullet}^\bullet \mathcal{O}^\bullet)$. It follows from the above observations and multiplicative properties of the spectral sequence that $d^{0,q}_1(f \cdot s) = df \cdot s + f \cdot d^{0,q}_1(s)$. Hence $d^{0,q}_1$ is a connection. Moreover $d^{0,q}_1(\omega \cdot s) = d\omega \cdot s + (-1)^p \omega \cdot d^{0,q}_1(s)$ if $\omega \in \Omega_{X \times X}$. Hence $d^{0,q}_1$ is deduced formally, as a differential of a twisted De Rham complex from the connection $d^{0,q}_1$. Hence the fact $d^{0,q}_1 \circ d^{0,q}_1 = 0$ implies that the connection is integrable.

The connection $d^{0,q}_1$ is the connecting homomorphism of the exact sequence

$$0 \to \Omega^{n-1} \otimes \mathcal{O}_{X^{\partial[1]}}^\bullet (p^\bullet)^* \Omega_{X^{\partial[1]}}^1 \to G^0/G^2 \to$$

$$\Omega^* \otimes \mathcal{O}_{X^{\partial[1]}}^\bullet (p^\bullet)^* \mathcal{O}_{X^{\partial[1]}}^\bullet \to 0.$$

Observe that an exact sequence

$$0 \to F^{n-1} \Omega^{n-1} \otimes \mathcal{O}_{X^{\partial[1]}}^\bullet (p^\bullet)^* \Omega_{X^{\partial[1]}}^1 \to F^n(G^0/G^2) \to$$

$$F^n \Omega^* \otimes \mathcal{O}_{X^{\partial[1]}}^\bullet (p^\bullet)^* \mathcal{O}_{X^{\partial[1]}}^\bullet \to 0.$$
if $X$ is a smooth projective algebraic variety over $\text{Spec} \, k$.

Assume that $X$ is a smooth holomorphic variety. Let $\Omega^{\otimes (n+2)} := \ker(\nabla : \mathcal{P}^{\otimes (n+2)} \to | \Omega^1_{X,\Delta[1]}(\mathcal{P}^{\otimes (n+2)})$). The seaf $\Omega^{\otimes (n+2)}$ is a locally constant sheaf on $X^{\Delta[1]}$ i.e. each $\Omega^{\otimes (n+2)}$ is locally constant on $X^{n+2}$. The complex $\Omega^{\otimes \Delta[1]}(\mathcal{P}^{\otimes (n+2)})$ is a resolution of $\Omega^{\otimes (n+2)}$. Hence the spectral sequence of $tR(p^\bullet)_*$ associated with the filtration $\{G^i\}$ converges to $H^*(tR(p^\bullet)_* \Omega^{\otimes (n+2)})$. The complex $(*q)$ is exact but at a zero term. Hence $\ker^{0,q} = H^q(tR(p^\bullet)_* \Omega^{\otimes (n+2)})$.

3.7. Let $\tilde{X}$ be a smooth proper scheme of finite type over $\text{Spec} \, k$ and let $D$ be a divisor with normal crossings in $\tilde{X}$. Let $X = \tilde{X} \setminus D$. Let $(\tilde{\mathcal{P}} \to \tilde{X}, \tilde{\nabla} : \tilde{\mathcal{P}} \to \Omega^1_{X}(\mathcal{D}) \otimes \mathcal{O}_{\tilde{X}} \tilde{\mathcal{P}})$ be the canonical extension of $(\mathcal{P} \to X, \nabla : \mathcal{P} \to \Omega^1_{X} \otimes \mathcal{O}_{X} \mathcal{P})$.

Consider the following morphisms between cosimplicial schemes

$$
\begin{align*}
X^{\Delta[1]} \xrightarrow{p^\bullet} (\tilde{X}^{\Delta[1]} =: \tilde{\mathcal{X}}) \\
\downarrow p^\bullet \quad \downarrow \tilde{p}^\bullet \\
(X \times X)^\bullet \xrightarrow{\tilde{p}^\bullet} (\tilde{X} \times \tilde{X})^\bullet
\end{align*}
$$

where the horizontal morphisms are induced by inclusion $X \hookrightarrow \tilde{X}$.

Let $D(n) := X \times \tilde{X}^n \times X \setminus X \times X^n \times X$ and $Y(n) := X \times \tilde{X}^n \times \tilde{X} \setminus X \times \tilde{X}^n \times X$. Let us set $\mathcal{X} := X^{\Delta[1]}$. The twisted logarithmic De Rham complex on $\mathcal{X}$,

$$
\Omega^\ast_{\mathcal{X}}(D(\bullet) + Y(\bullet))(\tilde{\mathcal{P}}^{\otimes (n+2)}) \quad \text{(at degree $n$ we have $\Omega^\ast_{\mathcal{X}^{n+2}}(D(n) + Y(n))(\tilde{\mathcal{P}}^{\otimes (n+1)})$)}
$$

and the twisted logarithmic relative De Rham complex on $\mathcal{X}$,

$$
\Omega^\ast_{\tilde{X}}(D(\bullet) + Y(\bullet)) := \Omega^\ast_{\tilde{X}/X}(D(\bullet) + Y(\bullet))(\tilde{\mathcal{P}}^{\otimes (n+2)})
$$

(at degree $n$ we have $\Omega^\ast_{\tilde{X} \times \tilde{X}^n \times \tilde{X}/y \times \tilde{X}}(D(n) + Y(n)) \otimes \mathcal{O}_{\tilde{X} \times \tilde{X}^n \times \tilde{X}}(\tilde{\mathcal{P}}^{\otimes (n+2)})$) are complexes of sheaves on $\mathcal{X}$. Let us set

$$
\mathcal{G}^i_n(\Omega^\ast_{\mathcal{X}^{n+2}}(D(n) + Y(n))(\tilde{\mathcal{P}}^{\otimes (n+2)})) := \text{image}(\Omega_{\mathcal{X}^{n+2}}^{\ast-1}(D(n) + Y(n))(\tilde{\mathcal{P}}^{\otimes (n+2)}) \otimes (p^n)^\ast \Omega^i_{\tilde{X} \times \tilde{X}}(\mathcal{D}) \to \Omega^\ast_{\mathcal{X}^{n+2}}(D(n) + Y(n))(\tilde{\mathcal{P}}^{\otimes (n+2)})).
$$

As before we construct a spectral sequence of the functor $tR(p^\bullet)_* \Omega^\ast$ associated with the filtration $\{\mathcal{G}^i\}$, and we get $d^p_{1,q} : \mathcal{E}^p_{1,q-1} \to \mathcal{E}^p_{1,q+1}$ and $\mathcal{E}^p_{1,q} = \Omega^p_{\tilde{X} \times \tilde{X}}(\mathcal{D}) \otimes \mathcal{O}_{\tilde{X} \times \tilde{X}} H^q(tR(p^\bullet)_* \Omega^\ast(D(\bullet) + Y(\bullet)))$. The differential $d^1_{0,q} \to \mathcal{E}^1_{0,q}$ is an integrable connection on $H^q(tR(p^\bullet)_* \Omega^\ast(D(\bullet) + Y(\bullet)))$.

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Theorem 3.7.1. We have

i) The sheaf \( H^q(tR\mathcal{p}_*\Omega^* \langle D(\bullet) + Y(\bullet) \rangle) \) is an extension of the sheaf \( H^q(tR\mathcal{p}_*\Omega^*) \).

ii) The connection
\[ \tilde{d}^1_{1,q} : H^q(tR\mathcal{p}_*\Omega^*(D(\bullet) + Y(\bullet))) \to \Omega^1_{\bar{X} \times X} \otimes \mathcal{O}_{\bar{X} \times X} H^q(tR\mathcal{p}_*\Omega^*(D(\bullet) + Y(\bullet))) \]

is an extension of the connection
\[ d_1^{0,q} : H^q(tR\mathcal{p}_*\Omega^*) \to \Omega^1_{X \times X} \otimes \mathcal{O}_{X \times X} H^q(tR\mathcal{p}_*\Omega^*). \]

Proof. These two statements are obvious because on \( X \), \( \Omega^*_X(D) = \Omega^*_X \) for an open subset \( \mathcal{U} \) of \( X \).

§4. Canonical cocycle associated to a variation of mixed Hodge structures.

4.0. Let \( X \) be a smooth complex variety and let \( \gamma \) be a smooth path on \( X \) from \( x_0 \) to \( z \).

Let \( M_1(x), M_2(x) \in \Omega^1(X) \otimes M_n(C) \). Then \( \int_{\gamma} M_1(x), M_2(x) \) is an iterated integral such that \( d(\int_{\gamma} M_1(x), M_2(x)) = M_1(x) \circ \int_{\gamma} M_2(x). \)

Let us choose \( L(x) \) and \( C(x) \) in \( \Omega^1(X) \otimes M_n(C) \). Assume that the matrix function \( \varphi_\gamma(z) \) satisfies a differential equation
\[ d\varphi_\gamma(z) + L(z)\varphi_\gamma(z) = 0, \ \varphi_\gamma(x_0) = Id. \]

Then the sum of iterated integrals
\[ F_\gamma(z) := \varphi_\gamma(z)(Id + \int_{\gamma} \varphi_\gamma(x)^{-1} \circ (-C(x)) \circ \varphi_\gamma(x) + \int_{\gamma} \varphi_\gamma(x)^{-1} \circ (-C(x)) \circ \varphi_\gamma(x) \varphi_\gamma(x)^{-1} \circ (-C(x)) \circ \varphi_\gamma(x) + \ldots \]
satisfies a differential equation
\[ dF_\gamma(z) + (L(z) + C(z))F_\gamma(z) = 0, \ \ F_\gamma(x_0) = Id. \]
Let us set
\[ f_0(z) := \varphi_\gamma(z), \quad f_1(z) := \varphi_\gamma(z) \int_\gamma \varphi_\gamma(x)^{-1} \circ (-C(x)) \circ \varphi_\gamma(x), \]
\[ \ldots \]
\[ f_n(z) := \varphi_\gamma(z) \int_\gamma \varphi_\gamma(x)^{-1} \circ (-C(x)) \circ \varphi_\gamma(x), \ldots. \]

Let \( g_k(z) = \sum_{i=0}^{k} f_i(z) \). The functions \( g_k(z) \) satisfy the following system of differential equations
\[ dg_0(z) + L(z)g_0(z) = 0, \quad g_0(x_0) = Id; \]
\[ dg_1(z) + L(z)g_1(z) + C(z)g_0(z) = 0, \quad g_1(x_0) = Id; \]
\[ \ldots \]
\[ dg_n(z) + L(z)g_n(z) + C(z)g_{n-1}(z) = 0, \quad g_n(x_0) = Id. \]

Assume that \( f_i(z) = 0 \) for \( i \geq n + 1 \). Then \( g_n = g_{n+1} \) and
\[ dg_n(z) + (L(z) + C(z))g_n(z) = 0. \]

Hence the columns of \( g_n(z) \) are flat sections of the connection
\[ \nabla : \mathcal{O}_X^n \to \Omega_X^1 \otimes \mathcal{O}_X^n \]
given by \( \nabla(s) = ds + (L + C)s \).

Let \( \gamma_1 \) be a path from \( z \) to \( z' \). Then we have
\[ F_{\gamma_1 \circ \gamma}(z') = F_\gamma(z') \cdot F_\gamma(z). \]

For an iterated integral of length \( n \) we get
\[ \varphi_{\gamma_1 \cdot \gamma}(z') \int_{\gamma_1 \cdot \gamma} (\varphi_{\gamma_1 \cdot \gamma}(x))^{-1} \circ (-C_n(x)) \circ \varphi_{\gamma_1 \cdot \gamma}(x), \ldots, (\varphi_{\gamma_1 \cdot \gamma}(x))^{-1} \circ (-C_1(x)) \circ \varphi_{\gamma_1 \cdot \gamma}(x) = \]
\[
\varphi_{\gamma_1}(z') \circ \varphi_\gamma(z) \left( \sum_{i=0}^{n} \left( \int_{\gamma_1} \left( \varphi_\gamma(z) \right)^{-1} \circ (\varphi_{\gamma_1}(x))^{-1} \circ (-C_n(x)) \circ \varphi_{\gamma_1}(x) \circ \varphi_\gamma(z) \right) \right),
\]

We suppose for simplicity that the principal \( G \)-bundle equipped with the integrable connection \( \mathcal{M} : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} \) was motivated by the last equality.

4.1. We suppose for simplicity that the principal \( G \)-bundle \( \pi : P \to X \) is trivial.

Let \( \pi : X \times G \to X \) be a principal \( G \)-bundle equipped with the integrable connection given by a one-form \( \omega \in \Omega^1(X) \otimes \mathfrak{g} \), where \( \mathfrak{g} = \text{Lie}(G) \).

Let \( \rho : G \to \text{Aut}_{\text{alg}}(\mathcal{O}[G]) \) be given by \( g(f) := g \cdot f \), where \( (g \cdot f)(x) := f(g^{-1} \cdot x) \). Let \( \tilde{\rho} : \mathfrak{g} \to \text{Der}_{\text{alg}}(\mathcal{O}[G]) \) be the induced map of Lie algebras.

**Lemma 4.1.1.** Let \( \mu : \mathcal{O}[G] \to \mathcal{O}[G] \otimes \mathcal{O}[G] \) be induced by the multiplication \( G \times G \to G, (g_2, g_1) \to g_2 \cdot g_1 \). Let us set \( \mu(f)(g_2, g_1) = \sum_i \mu(f)^i_2(g_2) \otimes \mu(f)^i_1(g_1) \). Let \( X \in \mathfrak{g} \). Then we have

\[
\mu(\tilde{\rho}(X)(f)) = \sum_i \tilde{\rho}(X)(\mu(f)^i_2) \otimes \mu(f)^i_1.
\]

**Proof.** It follows from the equality \( \mu(\rho f) = \sum_i \rho(\mu(f)^i_2) \otimes \mu(f)^i_1 \).

Let \( \varphi : G \to \text{Aut}(V) \) be a representation of \( G \) and let \( \tilde{\varphi} : \mathfrak{g} \to \text{End}(V) \) be the induced representation of the Lie algebra \( \mathfrak{g} \). Let us fix a base \( e_1, \ldots, e_n \) of \( V \). Let \( \varphi(g) = (a_{ij}(g))_{i,j} \) in this base. Let us set:

\[
\tau_\varphi := (a_{ij})_{i,j} \in \mathcal{O}[G] \otimes \text{End}(V).
\]

The group \( G \) acts on \( \mathcal{O}[G] \), hence it acts also on \( \mathcal{O}[G] \otimes \text{End}(V) \). We have

\[
g : (a_{i,j})_{i,j} \to (a_{i,j}(g^{-1}))_{i,j} \cdot (a_{i,j})_{i,j} = (\sum_k a_{i,k}(g^{-1})a_{k,j})_{i,j}.
\]

The subspace \( \tau_\varphi^{-1}\text{End}(V) \tau_\varphi \subset \mathcal{O}[G] \otimes \text{End}(V) \) is \( G \)-invariant.

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Lemma 4.1.2. Let $X \in \mathfrak{g}$ and $C \in \text{End}(V)$. We have

$$\hat{\rho}(X)(\tau_{\varphi}^{-1}C \tau_{\varphi}) = \tau_{\varphi}^{-1}[\varphi(X), C] \tau_{\varphi},$$

$$\hat{\rho}(X)(\tau_{\varphi}^{-1}C) = \tau_{\varphi}^{-1}\hat{\varphi}(X)C, \quad \hat{\rho}(X)(C \tau_{\varphi}) = -C \hat{\varphi}(X) \tau_{\varphi}.$$  

Proof. Observe that the subspaces $\tau_{\varphi}^{-1}\text{End}(V)$ and $\text{End}(V)\tau_{\varphi}$ of $\mathcal{O}[G] \otimes \text{End}(V)$ are also $G$-invariant. The group $G$ acts on $\tau_{\varphi}^{-1}\text{End}(V)$. The element $g \in G$ maps $(a_{i,j})_{i,j}^{-1} \circ C$ into $(a_{i,j})_{i,j}^{-1} \circ (a_{k,l}(g))_{k,l} \circ C$. Let $X \in \mathfrak{g}$. Then $\hat{\rho}(X)((a_{i,j})_{i,j}^{-1} \circ C) = \lim_{t \to 0} \frac{1}{t} ((a_{i,j})_{i,j}^{-1} \circ \exp tX)_{i,j} \circ C - (a_{i,j})_{i,j}^{-1} \circ C) = (a_{i,j})_{i,j}^{-1} \circ \varphi(X) \circ C.$

4.2. The associated vector bundle $(X \times G) \times_{\rho} \mathcal{O}[G] \rightarrow X$

equipped with the connection given by a one-form $\hat{\rho}(\omega) \in \Omega^1(X) \otimes \text{Der}_{\text{algebra}}(\mathcal{O}[G])$ is the pair $(\mathcal{P} = \pi_* \mathcal{O}_P \rightarrow X, \nabla)$ from section 3.1. Let $f \in \mathcal{O}(X)$ and $\chi \in \mathcal{O}[G]$. Then

4.2.0. $$\nabla (f \otimes \chi) = df \otimes \chi + f \hat{\rho}(\omega)(\chi).$$

The connection $\nabla$ defines trivially a connection also denoted by $\nabla$, on the vector bundle

$$(X \times G) \times_{\rho} \mathcal{O}[G] \otimes \text{End}(V) \rightarrow X.$$  

Lemma 4.2.1. Let $C(z) \in \mathcal{O}(X) \otimes \text{End}(V)$. We have

$$\nabla(\tau_{\varphi}^{-1}C(z) \tau_{\varphi}) = \tau_{\varphi}^{-1}(dC(z) + [\varphi(\omega), C(z)]) \tau_{\varphi}.$$  

Proof. The lemma follows from the formula 4.2.0 and Lemma 4.1.2.

4.3. Let $\mathcal{V}$ be a vector bundle on $X$ equipped with the integrable connection $\nabla_{\mathcal{V}} : \mathcal{V} \rightarrow \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{V}$. We shall assume that $\mathcal{V}$ is trivial and that it is equipped with the increasing filtration $\{W_i\}_i$ compatible with the connection, by trivial sub-bundles. We assume the associate graded bundle with the induced connection is a bundle

$$(X \times G) \times_{\varphi} \mathcal{V} \rightarrow X,$$
for a certain representation \( \varphi : G \to \text{Aut} V \), equipped with a connection given by a one-form

\[ L(z) := \nabla \varphi(\omega) \in \Omega^1(X) \otimes \text{End}(V). \]

Hence the connection \( \nabla_V \) is given by a one-form

\[ L(z) + C(z) \in \Omega^1(X) \otimes \text{End}(V). \]

Observe that \( C(z)^n = 0 \) if the filtration of has a length \( n \). We have

\[ dL(z) + L(z) \wedge L(z) = 0. \]

The integrability of the connection \( \nabla_V \) implies

\[ 4.3.1. \quad dC(z) + C(z) \wedge L(z) + L(z) \wedge C(z) + C(z) \wedge C(z) = 0. \]

**4.4.** Let \( X \) be a smooth holomorphic variety and let \( \Omega^*(X) \) be the De Rham complex of smooth complex values differential forms on \( X \) or let \( X \) be an affine algebraic variety over \( k \) and let \( \Omega^*(X) \) be the algebraic De Rham complex on \( X \).

We recall from 2.5 that \( T_{x,x} \) is the total complex of the double complex

\[
\bigoplus_{n=0}^{\infty} \mathcal{O}[G] \otimes \Omega^*(\mathcal{P})(X) \otimes \mathcal{O}[G] = \bigoplus_{n=0}^{\infty} \mathcal{O}[G] \otimes (\Omega^*(X) \otimes \mathcal{O}[G]) \otimes \mathcal{O}[G].
\]

**Definition 4.4.1.** Let us set

\[
\tau^{(n)} := \tau \varphi(g_n)^{-1} \circ (-C(z_n)) \circ \tau \varphi(g_n) \otimes \ldots \otimes \tau \varphi(g_1)^{-1} \circ (-C(z_1)) \circ \tau \varphi(g_1) \otimes \tau \varphi(g_0)^{-1} \in (\mathcal{O}[G] \otimes (\Omega^*(X) \otimes \mathcal{O}[G])) \otimes \mathcal{O}[G] \otimes \text{End}(V)
\]

and

\[
\tau_V = \tau_{\nabla_V} := \tau^{(0)} + \tau^{(1)} + \ldots + \tau^{(n)} + \ldots.
\]

Observe that only a finite number of \( \tau^{(i)} \) are non-zero, hence \( \tau_{\nabla_V} \in T_{x,x}^0 \otimes \text{End}(V) \).

**Lemma 4.4.2.** The element \( \tau_{\nabla_V} \) is a cocycle.

**Proof.** We have \((-1)^n \delta(\tau^{(n)}) = (-1)^n \sum_{i=1}^{n+1} (-1)^i \otimes \tau \varphi(g_i)^{-1} \circ (C(z_i) \wedge C(z_i)) \circ \tau \varphi(g_i) \otimes \tau \varphi(g_i)^{-1} \circ (-C(z_i)) \circ \tau \varphi(g_i) \otimes \tau \varphi(g_0)^{-1} \)

\[ \tau \varphi(g_{i-1})^{-1} \circ (-C(z_{i-1})) \circ \tau \varphi(g_{i-1}) \otimes \ldots \otimes \tau \varphi(g_1)^{-1} \circ (-C(z_1)) \circ \tau \varphi(g_1) \otimes \tau \varphi(g_0)^{-1} \]

and

\[
\delta(\tau^{(n-1)}) = \sum_{i=1}^{n-1} (-1)^{n-i-1} \otimes \tau \varphi(g_i)^{-1} \circ (-dC(z_i) - L(z_i) \circ C(z_i) - C(z_i) \circ L(z_i)) \circ \tau \varphi(g_i) \otimes \ldots.
\]

Hence it follows from 4.3.1 that \( \tau_{\nabla_V} \) is a cocycle.
Lemma 4.4.3. We have

\[(id_{T^0_{x,x}} \otimes \tau_{\nabla_V}) \circ \tau_{\nabla_V} = (\mathcal{M} \otimes id_V) \circ \tau_{\nabla_V}.\]

Proof. We recall that for a trivial bundle \(\pi : X \times G \to X\) the comultiplication \(\mathcal{M} : T^0_{x,x} \to T^0_{x,x} \otimes T^0_{x,x}\) is given by

\[
\mathcal{M}(f_{n+1}(g_{n+1}) \otimes \omega_n f_n(g_n) \otimes \ldots \otimes \omega_1 f_1(g_1) \otimes f_0(g_0)) = \sum_{i=0}^{n} (f_{n+1}(g_{n+1}) \otimes \omega_n f_n(g_n) \otimes \ldots \otimes \omega_{i+1} f_{i+1}(g_{i+1}) \otimes 1) \otimes (1 \otimes \omega_i f_i(g'_i g_i) \otimes \ldots \otimes \omega_1 f_1(g'_1 g_i) \otimes f_0(g'_0 g_i))
\]

Hence we have

\[
(\mathcal{M} \otimes id_V) \circ \tau^{(n)}(g_n, \ldots, g_1, g_0) = \sum_{i=0}^{n} (\tau_{\varphi}(g_n))^{-1} \circ (-C(z_n)) \circ \tau_{\varphi}(g_n) \otimes \ldots \otimes (\tau_{\varphi}(g_{i+1}))^{-1} \circ (-C(z_{i+1})) \circ \tau_{\varphi}(g_{i+1}) \otimes \ldots
\]

\[
\otimes (\tau_{\varphi}(g'_i g_i))^{-1} \circ (-C(z_i)) \circ \tau_{\varphi}(g'_i g_i) \otimes \tau_{\varphi}(g'_0 g_i)^{-1} = \sum_{i=0}^{n} \tau^{(n-i)}(g_n, \ldots, g_{i+1}, g_i) \otimes \tau^{(i)}(g'_1, \ldots, g'_i, g'_0).
\]

This implies the lemma.

Hence the class of \(\tau_{\nabla_V}\), which for simplicity, we also denote by \(\tau_{\nabla_V}\) is a representation of \(\mathcal{H}\). We shall show that \(\tau_{\nabla_V}\) is a mixed Hodge representation. First however we drop the assumption that the principal \(G\)-bundle \(\pi : P \to X\) and the vector bundle \(\mathcal{V}\) are trivial. We shall work in Cech cohomology.

4.5. Let \(\pi : P \to X\) be a principal \(G\)-bundle equipped with the integrable connection given by a one-form \(\omega \in \Omega^1(P) \otimes \mathfrak{g}\). Let \(\mathcal{U} = \{U_i\}_{i \in J}\) be an open covering of \(X\) and let \(\{\psi_i : \pi^{-1}(U_i) \to U_i \times G\}_{i \in J}\) be a family of \(G\)-isomorphisms. Let

\[
\{\psi_{j,i} : U_j \cap U_i \to G\}
\]

be the family of transition functions. They satisfy the cocycle condition

4.5.1. \(\psi_{k,i}(x) = \psi_{kj}(x) \cdot \psi_{ji}(x)\)
on $U_k \cap U_j \cap U_i$. For each $i$, let $\sigma_i : U_i \to P$ be a section on $U_i$ defined by $\sigma_i(x) = \psi_i^{-1}(x, e)$. Let $\theta$ be the left invariant $\mathfrak{g}$-valued canonical 1-form on $G$.

For each $i$, we define a $\mathfrak{g}$-valued one-form $\omega_i$ on $U_i$ by $\omega_i := (\sigma_i)^* \omega$. The forms $\{\omega_i\}_{i \in J}$ satisfy

4.5.2. \[ \omega_j = \text{ad}(\psi_{ij}^{-1}) \omega_i + \psi_{ij}^* \theta \]
on $U_i \cap U_j$ and they determine the connection form $\omega$.

Remark 4.5.3. Observe that the description of a principal $G$-bundle equipped with a connection as a family of transition functions satisfying cocycle condition and a family of one-forms satisfying 4.5.2 is valid in étale topology or flat topology.

4.6. Let $\rho : G \to \text{Aut}_{algebra}(\mathcal{O}[G])$ be given by $g(f) := \rho g f$, where $(\rho g f)(x) := f(g^{-1}x)$. Then $\{\Psi_{ji} := \rho \circ \psi_{ji}\}_{ji}$ are transition functions of the associated vector bundle

$$\mathcal{P} := (P \times_G \mathcal{O}[G] \to X).$$

Let $\tilde{\rho} : \mathfrak{g} \to \text{Der}_{algebra}(\mathcal{O}[G])$ be the induced map of Lie algebras. Let us set

$$\Lambda_i := \tilde{\rho}(\omega_i) \in \Omega^1(U_i) \otimes \text{Der}_{algebra}(\mathcal{O}[G]).$$

Then it follows from 4.5.2 that

4.6.1. \[ \Psi_{ji} \Lambda_i = \Lambda_j \Psi_{ji} + d\Psi_{ji}. \]

Hence the family of one-forms $\{\Lambda_i\}_{i \in J}$ defines an integrable multiplicative connection on $\mathcal{P}$. This is the connection $\nabla$ from section 3.1.

4.7. Let $\varphi : G \to \text{Aut}V$ be a representation of $G$. The functions $\{A_{ji} := \varphi \circ \psi_{ji}\}_{ji}$ satisfy the cocycle condition. They are transition functions of the associated vector bundle $\mathcal{L} := (P \times_{\varphi} V \to X)$. Let $\tilde{\varphi} : \mathfrak{g} \to \text{End}(V)$ be the induced map of Lie algebras. Let us set

$$\lambda_i := \tilde{\varphi}(\omega_i) \in \Omega^1(U_i) \otimes \text{End}(V).$$

It follows from 4.5.2 that

4.7.1. \[ A_{ji} \lambda_i = \lambda_j A_{ji} + dA_{ji}. \]
The one-forms \( \{\lambda_i\}_{i \in J} \) determine an integrable connection \( \nabla_L \) on \( L \).

4.7.2. We assume that \( L \) is a graded vector bundle and the connection \( \nabla_L \) is compatible with the gradation i.e.

\[ (L, \nabla_L) = (\oplus_\alpha L_\alpha, \oplus_\alpha \nabla_{L_\alpha}). \]

The transition functions \( \{A_{ji}\} \) and the one-forms \( \{\omega_i\} \) are compatible with the gradation.

4.8. Let \( \chi_{ji} : U_j \cap U_i \to \text{Aut}(V) \) \( ((j, i) \in J \times J) \) be a family of functions satisfying a cocycle condition. Let \( M_i \in \Omega^1(U_i) \otimes \text{End}(V) \) \( (i \in J) \) be a family of one-forms such that

4.8.1. \( \chi_{ji} \circ M_i = M_j \circ \chi_{ji} + d\chi_{ji} \).

The cocycle \( \{\chi_{ji}\} \) and the family of one-forms \( \{M_i\} \) define a vector bundle \( V \) and a connection \( \nabla_V \). We assume that the connection \( \nabla_V \) is integrable.

We assume that the vector bundle \( V \) is equipped with the filtration \( \{W_\alpha V\}_\alpha \) compatible with the transition functions \( \{\chi_{ji}\} \) and the connection one-forms \( \{M_i\} \). We assume further that the associated graded vector bundle with the induced connection is \( (\oplus_\alpha L_\alpha, \oplus_\alpha \nabla_{L_\alpha}) \).

In terms of transition functions and connection matrices it means:

\[ \chi_{ji} = A_{ji} + B_{ji}, \quad M_i = \lambda_i + C_i \]

and \( B_{ji} \) and \( C_i \) vanish when passing to the associated graded vector bundle.

4.9. We recall that a Cech complex \( C^*(\mathcal{U}; \Omega^*(\mathcal{P})) \) associated with a covering \( \mathcal{U} = \{U_i\}_{i \in J} \) is defined in the following way:

\[ C^m(\mathcal{U}; \Omega^n(\mathcal{P})) := \{f(i_0, \ldots, i_m) \in \Omega^n(\mathcal{P})(U_{i_0} \cap \ldots \cap U_{i_m})\}_{(i_0, \ldots, i_m) \in J^m}; \]

\[ \{(Df)(i_0, \ldots, i_{m+1})\} := \{\sum_{k=0}^{m+1} (-1)^k f(i_0, \ldots, \hat{i}_k, \ldots, i_{m+1})\}; \]

\( \partial \) is induced by the differential of twisted De Rham complex;

\[ C^*(\mathcal{U}; \Omega^*(\mathcal{P})) := \text{Tot}(\bigoplus_{n,m=0}^{\infty} C^m(\mathcal{U}; \Omega^n(\mathcal{P}))) \]

with a differential \( d \) equal \( D + (-1)^m \partial \) on \( C^m(\mathcal{U}; \Omega^n(\mathcal{P})) \).
The trivialization \( \{ \psi_i : \pi^{-1}(U_i) \to U_i \times G \}_{i \in J} \) of the principal \( G \)-bundle \( \pi : P \to X \) induces an isomorphism

\[
\mathcal{O}_X(U_i) \otimes \mathcal{O}[G] \cong \Omega^0(\mathcal{P})(U_i).
\]

We define elements \( \tau_{\phi_i} \in \Omega^0(\mathcal{P})(U_i) \otimes \text{End}(V) \) by setting

\[
\tau_{\phi_i} := 1 \otimes \tau_{\phi} \in \mathcal{O}_X(U_i) \otimes \mathcal{O}[G] \otimes \text{End}(V).
\]

Let us define an element \( \sigma(\nabla_V) \) in

\[
C^1(\mathcal{U}; \Omega^0(\mathcal{P})) \otimes \text{End}(V) \oplus C^0(\mathcal{U}; \Omega^1(\mathcal{P})) \otimes \text{End}(V)
\]

by the following formula

\[
\sigma(\nabla_V) = \sigma := b + c := \{(j, i) \to \tau_{\phi_j}^{-1} \circ (-B_{ji}) \circ \tau_{\phi_i} \in \Omega^0(\mathcal{P})(U_j \cap U_i) \otimes \text{End}(V)\} + \{(i) \to \tau_{\phi_i}^{-1} \circ (-C_i) \circ \tau_{\phi_i} \in \Omega^1(\mathcal{P})(U_i) \otimes \text{End}(V)\}.
\]

**Lemma 4.9.1.** We have

\[
\partial c - c \wedge c = 0 \text{ in } C^0(\mathcal{U}; \Omega^2(\mathcal{P})) \otimes \text{End}(V);
\]

\[
D b - b \cup b = 0 \text{ in } C^2(\mathcal{U}; \Omega^0(\mathcal{P})) \otimes \text{End}(V);
\]

\[
\partial b + c \cup b + b \cup c - Dc = 0 \text{ in } C^2(\mathcal{U}; \Omega^0(\mathcal{P})) \otimes \text{End}(V).
\]

**Proof.** The first equality follows from the fact that the connections \( \nabla_L \) and \( \nabla_V \) are integrable. The fact that \( \{ \chi_{ji} \} \) is a cocycle implies the second equality. The third one follows from the formula 4.8.1.
Lemma 4.9.2. We have

\[ d\sigma(\nabla_V) = \sigma(\nabla_V) \cup \sigma(\nabla_V) \text{ in } C^n(\mathcal{U}; \Omega^n(\mathcal{P})) \otimes \text{End}(V). \]

Proof. The lemma follows from Lemma 4.9.1.

Let \( \chi'_{ji} : U_j \cap U_i \to \text{Aut}(V') \) be another family of functions satisfying a cocycle condition and let \( M'_i \in \Omega^1(U_i) \otimes \text{End}(V') \) be a family of 1-forms satisfying 4.8.1. Assume that the vector bundle \( V' \) determined by the cocyle \( \chi'_{ji} \) and the connection \( \nabla_V \) defined by the family \( \{M'_i\} \) satisfy the condition 4.8. In terms of transition functions and connection matrices it means

\[ \chi'_{ji} = A'_{ji} + B'_{ji}, \quad M'_i = \lambda'_i + C'_i \]

and \( B'_{ij} \) and \( C'_i \) vanish when passing to the associated graded vector bundle. The associated graded bundle \( L' \) is \( P \times_{\varphi'} V' \to X \), where \( \varphi' : G \to \text{Aut}(V') \) is a representation of \( G \), \( A'_{ji} = \varphi' \circ \psi_{ij} \) and \( \lambda'_i = \dot{\varphi}'(\omega_i) \).

We assume that the triples \( (V, \{W_\alpha V\}_\alpha, \nabla_V) \) and \( (V', \{W_\alpha V'\}_\alpha, \nabla_{V'}) \) are isomorphic. Hence we have an isomorphism

\[ (L, \nabla_L) \approx (L', \nabla_{L'}). \]

Let \( f : V \to V' \) be an isomorphism of fibers over \( x \). The fact that the image of the monodromy representation \( \Theta_x : \pi_1(X, x) \to G \) is Zariski dense in \( G \) implies that

\[ \varphi'(g) = f \circ \varphi(g) \circ f^{-1} \]

for all \( g \in G \).

Locally the isomorphism \( (V, \{W_\alpha V\}_\alpha, \nabla_V) \approx (V', \{W_\alpha V'\}_\alpha, \nabla_{V'}) \) means that there are \( h_i \in O_X(U_i) \otimes \text{Hom}(V', V) \) such that

\[ h_j \chi'_{ji} = \chi_{ji} h_i \quad \text{and} \quad h_i M'_i = M_i h_i + dh_i. \]

Let us set

\[ \chi''_{ji} = f^{-1} \chi'_{ji} f \]
and in general
\[ (f')'' = f^{-1}(f)''. \]

Observe that \( A_{ji}' = A_{ji} \) and \( \lambda_i'' = \lambda_i \). The cocycle \( \{\chi_{ji}'\} \) and a family \( \{M_i''\} \) define a vector bundle \( \mathcal{V}'' \) and an integrable connection \( \nabla_{\mathcal{V}''} \).

Let us set
\[ k_i := h_i \circ f \in \mathcal{O}_X(U_i) \otimes \text{End}(V). \]

We have
\[ k_j \chi_{ji}'' = \chi_{ji}k_i \text{ and } k_i M_i'' = M_i k_i + dk_i. \]

Hence the triples \( (\mathcal{V}, \{W_a\}, \nabla_{\mathcal{V}}) \) and \( (\mathcal{V}'', \{W_a\}'', \nabla_{\mathcal{V}''}) \) are isomorphic. Let
\[ \eta \in \mathcal{C}^0(\mathcal{U}; \Omega^0_X(\mathcal{P})) \otimes \text{End}(V) \]
be given by:
\[ \eta = \{(i) \to \tau_{\varphi_i}^{-1} \circ (k_i - \text{Id}) \circ \tau_{\varphi_i} \in \Omega^0_X(\mathcal{P})(U_i) \otimes \text{End}(V)\}. \]

**Lemma 4.9.3.** We have
\[ \sigma - \sigma'' = d\eta - \sigma \cup \eta + \eta \cup \sigma'', \text{ where } \sigma = \sigma_{\nabla_{\mathcal{V}}} \text{ and } \sigma'' = \sigma_{\nabla_{\mathcal{V}'}}. \]

**Lemma 4.9.4.** Let us set \( \sigma' = \sigma_{\nabla_{\mathcal{V}'}} \). We have \( \sigma' = f \circ \sigma'' \circ f^{-1} \).

**4.10.** Now we shall compute Cech cohomology of \( \mathcal{V}_{yx} := (X; x, y)^{(\Delta[1]:0,1)} \). Let
\[ \partial : \mathcal{O}[G] \otimes C^*(\mathcal{U}; \Omega^*_{\mathcal{P}}) \otimes \mathcal{O}[G] \to \mathcal{O}[G] \otimes C^*(\mathcal{U}; \Omega^*_{\mathcal{P}}) \otimes \mathcal{O}[G] \]
be the differential of a tensor product of complexes \( C^*(\mathcal{U}; \Omega^*_{\mathcal{P}}) \).

The two augmentations \( \varepsilon_x \) and \( \eta_x \) are non-zero only on \( \mathcal{C}^0(\mathcal{U}; \Omega^0(\mathcal{P})) = \prod_{i \in J} \mathcal{P}(U_i) \).

We choose \( i_0 \) such that \( x \in U_{i_0} \). The augmentation are compositions of the projection of \( \mathcal{C}^0(\mathcal{U}; \Omega^0(\mathcal{P})) \) onto \( \mathcal{P}(U_{i_0}) \) with \( \varepsilon_x \) and \( \eta_x \) defined in

Differentials
\[ \delta : \mathcal{O}[G] \otimes C^*(\mathcal{U}; \Omega^*_{\mathcal{P}}) \otimes \mathcal{O}[G] \to \mathcal{O}[G] \otimes C^*(\mathcal{U}; \Omega^*_{\mathcal{P}}) \otimes (n-1) \otimes \mathcal{O}[G] \]

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are defined using structure maps $\delta^i : y \times X^{n-1} \times x \to y \times X^n \times x$ $i = 0, 1, \ldots, n$ and augmentations. Let us set:

$$C^*(U; \mathcal{O}[G] \otimes \Omega_{y^n}^* \times \mathcal{O}[G]) := \operatorname{Tot}(\bigoplus_{n=0}^{\infty} \mathcal{O}[G] \otimes C^*(U; \Omega^* (\mathcal{P})^\otimes n \otimes \mathcal{O}[G])$$

with a differential $d$ obtain from $\partial$ and $\delta$ with the same sign convention as in 4.9.

**Definition 4.10.1.** Let us set:

$$\tau^{(n)}_{\nabla V} := \sigma(\nabla V)(g_n, z_n) \otimes \ldots \sigma(\nabla V)(g_1, z_1) \otimes \tau_(g_0)^{-1} \in C^*(U; \mathcal{O}[G] \otimes \mathcal{O}[G]) \otimes \operatorname{End}(V)$$
and

$$\tau_{\nabla V} := \tau^{(0)}_{\nabla V} + \tau^{(1)}_{\nabla V} + \ldots + \tau^{(n)}_{\nabla V} + \ldots \in \operatorname{Tot}(\bigoplus_{n=0}^{\infty} \mathcal{O}[G] \otimes C^*(U; \Omega^* (\mathcal{P})^\otimes n) \otimes \mathcal{O}[G]) \otimes \operatorname{End}(V).$$

**Lemma 4.10.2.** The element $\tau_{\nabla V}$ is a cocycle.

**Proof.** The lemma follows from Lemma 4.9.2.

Let us set

$$\zeta(j, i) := \sigma(g_j) \otimes \ldots \otimes \sigma(g_1) \otimes \eta(g) \otimes \sigma''(h_i) \otimes \ldots \otimes \sigma''(h_1) \otimes \tau_{\nabla V}(g_0).$$

Without lost of generality we can assume that the family $\{h_i\}$ is such that $h_{0o}(x) = Id$.

**Lemma 4.10.3.** We have

$$d \left( \sum_{i, j=0}^{\infty} (-1)^j \zeta(j, i) \right) = \tau_{\nabla V} - \tau_{\nabla V''}.$$

**Proof.** It follows from Lemma 4.9.3.

**Lemma 4.10.3.** We have

$$\tau_{\nabla V''} = f \circ \tau_{\nabla V} \circ f^{-1}.$$

**Proof.** It follows from Lemma 4.9.4.

4.11. Mixed Hodge representations.

**Definition 4.11.1.** Let $V$ be a vector space carring a mixed Hodge structure. Let $H$ be a Hopf algebra equipped with a mixed Hodge structure such that the structures maps are
morphism of mixed Hodge structures. We say that a representation \( \rho : V \to H \otimes V \) of the Hopf algebra \( H \) is a mixed Hodge representation if \( \rho \) is a morphism of mixed Hodge structures.

We shall show that the class \( \tau_\nabla : V \to \mathcal{H} \otimes V \) constructed in sections 4.4 (for trivial bundles) and 4.10 (in general case) is a mixed Hodge representation. First we consider the case when \( \pi : P \to X \) and the vector bundle \( V \) are trivial (see sections 4.1-4.4).

4.11.2. We assume that \( \mathcal{O}[G] \) and \( \mathcal{P} \) satisfy 3.2.2. We assume that the vector bundle \( V \) carry a variation of mixed Hodge structures such that the triple

\[ (\mathcal{V}, \{ W_i \}_i, \nabla_\mathcal{V} : \mathcal{V} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{V}) \]

is such as in 4.3. We assume that the representation \( \varphi : GrV = V \to \mathcal{O}[G] \otimes (GrV) = \mathcal{O}[G] \otimes V \) is a Hodge representation. \( V \) is a fiber of \( \mathcal{V} \) at \( x \) and it is canonically equal \( GrV = \oplus_i Gr^i_{\mathcal{V}} V \). This implies that the variation of Hodge structures \( Gr\mathcal{V} = \oplus_i Gr^i_{\mathcal{V}} \mathcal{V} \) is in \( \text{VHS}_\Theta (\Theta : \pi_1(X, x) \to G \) is the monodromy homomorphism of the principal \( G \)-bundle \( \pi : X \times G \to X \).

**Proposition 4.11.3.** Let \( X \) be a smooth projective complex variety. Let \( \mathcal{V} \) be a variation of mixed Hodge structures over \( X \) satisfying 4.11.2. Then the representation \( \tau_\nabla : V \to \mathcal{H} \otimes V \) of \( \mathcal{H} \) is a mixed Hodge representation.

**Proof.** The matrix \( C(z) \in \Omega^1(X) \otimes W_{-1} \text{End}(V) \) because \( C(z) \) is nilpotent with respect to the weight filtration of \( \mathcal{V} \). The representation \( \tau_\varphi \in \mathcal{O}[G] \otimes \text{End}(V) \) is a Hodge representation compatible with the filtration \( \{ W_i \}_i \) of \( V \), hence \( \tau_\varphi \in \mathcal{O}[G] \otimes W_0(\text{End}(V)) \). Therefore \( \tau^{(n)} \in W_n(T^0) \otimes W_{-n}(\text{End}(V)) \). This implies \( \tau_\nabla \in W_0(\mathcal{H} \otimes \text{End}(V)) \). Observe that \( \tau_\varphi \in F^0(\mathcal{O}[G] \otimes \text{End}(V)) \). The matrix \( C(z) \in \Omega^1(X) \otimes F^{-1}(\text{End}(V)) \) because the connection \( \nabla_\mathcal{V} \) satisfies \( \nabla_\mathcal{V}(F^p\mathcal{V}) \subset \Omega^1_X \otimes_{\mathcal{O}_X} (F^{p-1}\mathcal{V}) \). Hence \( \tau_\nabla \in F^0(\mathcal{H} \otimes \text{End}(V)) \).

Now we consider the general case. The condition: “the triple \( (\mathcal{V}, \{ W_i \}_i, \nabla_\mathcal{V} : \mathcal{V} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{V}) \) is such as in 4.3” we replace by “the triple \( (\mathcal{V}, \{ W_i \}_i, \nabla_\mathcal{V} : \mathcal{V} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{V}) \) is such as in 4.8.”

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Proposition 4.11.4. Let $X$ be a smooth projective complex variety. Let $\mathcal{V}$ be a variation of mixed Hodge structures over $X$ satisfying 4.11.2. Then the representation $\tau_{\mathcal{V}} : V \to \mathcal{H} \otimes V$ of $\mathcal{H}$ is a mixed Hodge representation.

**Proof.** Observe that $C_j \in \Omega^{\bullet}(U_j) \otimes W_{-1} \text{End}(V)$ and $B_{ji} \in \Omega^{\bullet}(U_j \cap U_i) \otimes W_{-1} \text{End}(V)$ because $C_j$ and $B_{ji}$ are nilpotent with respect to the weight filtration of $\mathcal{V}$.

4.12. The element $\tau_{\mathcal{V}} \in \mathcal{H} \otimes \text{End}(V)$. We recall that $\mathcal{H} \otimes \text{End}(V)$ is a fiber of $H^0(tR^p\Omega^*) \otimes \text{End}(V)$ over $(x,x) \in X \times X$. We shall calculate the monodromy of $\tau_{\mathcal{V}}$.

We construct a section $s$ of $H^0(tR^p\Omega^*) \otimes \text{End}(V)$ such that $s(x) = \tau_{\mathcal{V}}$ and next we find conditions when this section is flat.

We suppose first that the principal $G$-bundle $\pi : P \to X$ and the pair $(\mathcal{V},\{W_i\})$ are trivial. We recall that $\tau_{\mathcal{V}} = \tau^{(0)} + \tau^{(1)} + \ldots + \tau^{(n)}$ for some $n$ and we assume that $\tau^{(n)} \neq 0$. Let $K_0(z), \ldots, K_n(z) \in \mathcal{O}(X) \otimes \text{End}(V)$ and let $K_i(x) = \text{Id}$ for $i = 0, 1, \ldots, n$. Then

$$\tau(z) := \tau^{(0)} \circ K_n(z) + \tau^{(1)} \circ K_{n-1}(z) + \ldots + \tau^{(n)} \circ K_0(z)$$

is a section of $H^0(tR^p\Omega^*)$ such that $\tau(x) = \tau_{\mathcal{V}}$. This section is flat if and only if the functions $K_i(z)$ satisfy the following system of differential equations

$$dK_0(z) + L(z) \circ K_0(z) = 0,$$
$$dK_1(z) + L(z) \circ K_1(z) + C(z) \circ K_0(z) = 0,$$
$$\ldots$$
$$dK_n(z) + L(z) \circ K_n(z) + C(z) \circ K_{n-1}(z) = 0.$$

Observe that the functions $K_0(z), K_1(z), \ldots, K_n(z)$ coincide with sums of iterated integrals $g_0, g_1, \ldots, g_n$ from 4.0.

**Definition 4.12.1.** Let us set

$$\int_{\gamma} \tau_{\mathcal{V}} := \varphi_{\gamma}(z) + \varphi_{\gamma}(z) \circ \int_{\gamma} \varphi_{\gamma}(t)^{-1} \circ (-C(t)) \circ \varphi_{\gamma}(t) + \varphi_{\gamma}(z) \circ \int_{\gamma} \varphi_{\gamma}(t)^{-1} \circ (-C(t)) \circ \varphi_{\gamma}(t), \varphi_{\gamma}(t)^{-1} \circ (-C(t)) \circ \varphi_{\gamma}(t) + \ldots.$$ 

($\gamma$ is a path from $x$ to $z$, $\varphi_{\gamma}(z)$ is given by $d\varphi_{\gamma}(z) + L(z)\varphi_{\gamma}(z) = 0, \varphi_{\gamma}(x) = \text{Id}$).
Lemma 4.12.2. Let $\gamma \in \pi_1(X, x)$. Then the monodromy transformation of $\tau_{\nabla_V}$ along $\gamma$ is given by
\[ \gamma : \tau_{\nabla_V} \to \tau_{\nabla_V} \circ \int_\gamma. \]

Proof. It follows from equalities $K_i(z) = g_i(z)$.

Now we consider general case. Let $X' = (X, x)$. We shall calculate cohomology sheaves $H^i := H^i(\Omega^{\bullet}(P)) \otimes C^\bullet(U; \Omega^0(P))$. (We recall that $\Omega^* = \Omega^*_{X\times X} (P^{\otimes \bullet+1})$.) Next we compute the action of the Gauss-Manin connection on global sections of $H_i$.

Let us set
\[ T := \text{Tot}(\bigoplus_{n=0}^{\infty} C^*(U; \Omega^0(P)) \otimes C^*(U; \Omega^0(P))). \]

We have
\[ H^i(T) = \Gamma(X, H^i). \]

We define an element $\kappa \in C^0(U; \Omega^0(P)) \otimes \text{End}(V)$ in the following way:
\[ \kappa := ((i) \to \tau_{\varphi_i}^{-1} \circ K_i), \]

where $K_i \in \mathcal{O}_X(U_i) \otimes \text{End}(V)$ and $K_{i_0}(x) = \text{Id}_V$. Let us set
\[ K := \sum_{n=0}^{\infty} \sigma(g_n) \otimes \ldots \otimes \sigma(g_1) \otimes \kappa \in T \otimes \text{End}(V). \]

Lemma 4.12.3. The element $K$ is a global section of $H^0 \otimes \text{End}(V)$ if and only if
\[ B_{j,i}K_i + A_{j,i}K_i = K_j \]
for all pairs $(j, i)$.

Proof. One verifies that the element $K$ is a cocycle in $T$ if and only if $B_{j,i}K_i + A_{j,i}K_i = K_j$.

Observe that a complex
\[ T' := \text{Tot}(\bigoplus_{n=0}^{\infty} C^*(U; \Omega^0(P)^{\otimes n}) \otimes C^*(U; \Omega^0(P))) \]

computes the cohomology groups $H^i(tR\Omega^{\bullet}_X (P^{\otimes \bullet+1}))$. The section $s$ of $H^i$ (i.e., an element $s \in T'$ such that $ds = 0$ in $T$) is a flat section for the connection $d_{1,i}^{0,i} : H^i \to \Omega^1_{x \times X} \otimes \mathcal{O}_{x \times X} \mathcal{H}^i$ if and only if $ds = 0$ in the complex $T'$. Hence we get the following result.
Lemma 4.12.4. The (local) section $K$ is flat over $\bigcup_{i \in I} U_i$, $I \subset J$ if and only if

$$dK_i + \lambda_i K_i + C_i K_i = 0$$

for all $n$ and all $i \in I$.

Observe that $K$ is a flat section of $\mathcal{H}^0 \otimes \text{End}(V)$ if and only $\kappa$ is a flat section of the principal $GL(V)$-bundle corresponding to $V$. Let $\gamma$ be a path from $x$ to $z$. We denote by $\kappa(z) \in \text{End}(V)$ a value at $z$ of the flat section $\kappa$ continued along the path $\gamma$.

Definition 4.12.5. Let us set

$$\int_{\gamma} \tau_{\nabla_V} := \kappa(z).$$

Lemma 4.12.6. Let $\gamma \in \pi_1(X, x)$. The monodromy transformation of $\tau_{\nabla_V}$ along $\gamma$ is given by

$$\gamma : \tau_{\nabla_V} \rightarrow \tau_{\nabla_V} \circ \int_{\gamma} \tau_{\nabla_V}.$$ 

The representation $\tau_{\nabla_V} : V \rightarrow \mathcal{H} \otimes V$ of the Hopf algebra $\mathcal{H}$ induces a representation

$$(\tau_{\nabla_V})^* : (\text{Spec } \mathcal{H})(C) \rightarrow \text{End}(V)$$

given by

$$(\tau_{\nabla_V})^*(\sigma) := (\sigma \otimes \text{id}_V) \circ \tau,$$

where $\sigma \in (\text{Spec } \mathcal{H})(C) = Hom_{C-algebra}(\mathcal{H}, C)$. Let $\Theta_x : \pi_1(X, x) \rightarrow (\text{Spec } \mathcal{H})(C)$ be the monodromy representation of the principal $(\text{Spec } \mathcal{H})(C)$-bundle $\mathcal{H}^0$.

Lemma 4.12.7. We have

$$(\tau_{\nabla_V}) \circ \Theta_x = \int_{\gamma} \tau_{\nabla_V}.$$

Proof. Let $g = \Theta_x(\gamma)$ and let $\tau_{\nabla_V} = (f_{ij}(-)) \in \mathcal{H} \otimes \text{End}(V)$. We view $f_{ij}(-)$ as functions on $(\text{Spec } \mathcal{H})(C)$. After the monodromy transformation along the loop $\gamma \in \pi_1(X, x)$ the element $\tau_{\nabla_V}$ changes into $(f_{ij}(-g)) = (f_{ij}(-)) \circ (f_{ij}(g))$. Lemma 4.12.6 implies that $(f_{ij}(g)) = \int \gamma \tau_{\nabla_V}$. It follows from the definition of $(\tau_{\nabla_V})^*$ that $(\tau_{\nabla_V})^*(g) = (f_{ij}(g))$. Hence we proved the lemma.
Corollary 4.12.8. The monodromy homomorphism at $x$ of the connection $\nabla \mathcal{V} : \mathcal{V} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{V}$ is given by $\pi_1(X, x) \ni \gamma \to \int_\gamma \tau_{\nabla \mathcal{V}} \in \text{End}(\mathcal{V})$.

5. The Classification of variations of mixed Hodge structures.

5.0. Let $\mathcal{VMHS}_X(P)$ be a category of variations of mixed Hodge structures $\mathcal{V}$ over $X$ such that $\text{Gr}_{W} \mathcal{V}$ is in $\mathcal{VHS}_X(P)$. Let $\mathcal{MHRep}(\mathcal{H})$ be a category of mixed Hodge representations of $\mathcal{H}$. Let

$$F : \mathcal{VMHS}_X(P) \to \mathcal{MHRep}(\mathcal{H})$$

be a functor which to a variation of mixed Hodge structures $\mathcal{V}$ associates a mixed Hodge representation $\tau_{\mathcal{V}_x} : V_x \to \mathcal{H} \otimes V_x$, where $V_x$ is a fiber of $\mathcal{V}$ over $x \in X$. The functor $F$ on morphisms is a restriction to fibers over $x$. It follows from Lemmas 4.9.3, 4.9.4 and Propositions 4.11.3, 4.11.4 that $F$ is well defined. Our aim is to prove that $F$ is an equivalence of categories.

Theorem 5.0.1. Let $X$ be a smooth complex projective variety. Then the functor $F : \mathcal{VMHS}_X(P) \to \mathcal{MHRep}(\mathcal{H})$ is an equivalence of categories.

Proof. Let $\mathcal{V}$ and $\mathcal{V}'$ be in $\mathcal{VMHS}_X(P)$. First show that

$$F : \text{Hom}_{\mathcal{VMHS}_X(P)}(\mathcal{V}, \mathcal{V}') \to \text{Hom}_{\mathcal{MHRep}(\mathcal{H})}(V_x, V'_x)$$

is bijective. It follows from that the restriction to fibers over $x$,

$$G : \text{Hom}_{\mathcal{VMHS}_X(P)}(\mathcal{V}, \mathcal{V}') \to \text{Hom}_{\pi_1(X, x)}(V_x, V'_x)$$

is bijective. This implies that $F$ is injective. Let $\varphi : V_x \to V'_x$ be a morphism in $\mathcal{MHRep}(\mathcal{H})$. It follows from that $\varphi \in \text{Hom}_{\pi_1(X, x)}(V_x, V'_x)$. Hence there is $f : \mathcal{V} \to \mathcal{V}'$ such that $G(f) = \varphi$ and therefore also $F(f) = \varphi$.

It rests to show that any mixed Hodge representation $\tau : V \to \mathcal{H} \otimes V$ is isomorphic to $F(\mathcal{V})$ for a certain variation of mixed Hodge structures $\mathcal{V}$.

Let us set $\mathcal{V}_\tau := (\text{Spec} \mathcal{H}^0)(C) \times_{(\text{Spec} \mathcal{H})(C)} V$. $((\text{Spec} \mathcal{H})(C)$ acts on $V$ on the right by $(v, g) \to g^{-1}(v)$). Let $\tau' : \mathcal{H}^0 \otimes V \to \mathcal{H} \otimes (\mathcal{H}^0 \otimes V)$ be the diagonal action of $\mathcal{H}$. Let $\iota : \mathcal{H}^0 \otimes V \to \mathcal{H} \otimes (\mathcal{H}^0 \otimes V)$ be given by $\iota(w) = 1 \otimes w$. Then $V_\tau = (\mathcal{H}^0 \otimes V)^{(\text{Spec} \mathcal{H})(C)} = \ldots$
This implies that $V$ carries a variation of mixed Hodge structures because $\tau' - \iota$ is a morphism of variations of mixed Hodge structures. The monodromy representations at $x$ of the variation $V$ is equal to the composition

$$\pi_1(X, x) \xrightarrow{\Theta} (\text{Spec } \mathcal{H})^* \xrightarrow{\pi} \text{End}(V).$$

Let $\tau_{V_x}$ be a cocycle associated to $V$. Then the monodromy representation at $x$ of $V$ is equal $(\tau_{V_x})^* \circ \Theta_x$.

### 6. Classification of algebraic differential equations with regular singular points.

6.0. Let $X$ be a smooth complex algebraic variety, complement of a divisor with normal crossings in a smooth projective variety.

Let $P \rightarrow X$ be a holomorphic principal $G$-bundle equipped with a holomorphic integrable connection. We assume that the image of the monodromy homomorphism at $x$

$$\theta : \pi_1(X, x) \rightarrow G$$

is Zariski dense in $G$.

Let $DE(X; \theta)_{hol}$ be a category of holomorphic vector bundles $V$ on $X$ equipped with an increasing filtration $\{W_i V\}$ by holomorphic sub-vector bundles and with a holomorphic integrable connection $\nabla_V$ compatible with the filtration $\{W_i V\}$ such that the pair $(Gr_{W} V, \nabla_{Gr_{W} V})$ ($\nabla_{Gr_{W} V}$ is a connection induced by $\nabla_V$ on $Gr_{W} V$) is of the form $(P \times_{\rho} V; \nabla_{\rho})$, where $\rho : G \rightarrow \text{Aut}V$ is a representation and $\nabla_{\rho}$ is the connection induced on the associated vector bundle by the connection on the principal $G$-bundle $P \rightarrow X$.

We recall from previous sections that given a principal $G$-bundle $P \rightarrow X$ equipped with an integrable connection we have constructed a Hopf algebra $\mathcal{H}$ and a holomorphic bundle $\mathcal{H}^0$ equipped with an integrable holomorphic connection.

We denote by $\text{Rep}(\mathcal{H})$ the category of representations of the Hopf algebra $\mathcal{H}$ in a finite dimensional vector spaces.
Theorem 6.0.1. The categories $DE(X, \theta)_{hol}$, $\text{Rep}(\pi_1(X, x); \theta)$ and $\text{Rep}(\mathcal{H})$ are equivalent.

Proof. It follows from [D1] that the first two categories are equivalent. The equivalence

$$G : DE(X, \theta)_{hol} \to \text{Rep}(\pi_1(X, x); \theta)$$

associates to a pair $(\mathcal{V}, \nabla)$ a monodromy representation of $\pi_1(X, x)$ in a fiber over $x$ and to a morphism $f : (\mathcal{V}, \nabla) \to (\mathcal{V}', \nabla')$ its restriction to fibers over $x$, $f_x : V_x \to V'_x$. The functor $F$ is well defined by Lemmas 4.1.0.2. and 4.10.3. It is injective on morphisms because the functor $G$, which on morphisms is also a restriction to fibers over $x$, is an equivalence of categories. Let $(\mathcal{V}, \nabla)$ and $(\mathcal{V}', \nabla')$ be in $DE(X, \text{th})_{hol}$. Let $\alpha \in \text{Hom}_{\text{Rep}(\mathcal{H})}(\pi_1(\mathcal{V}, \nabla), (\mathcal{V}', \nabla'))$. The monodromy representations of $(\mathcal{V}, \nabla)$ and $(\mathcal{V}', \nabla')$ factor through $(\pi_1(\mathcal{V}, \nabla))^*$ and $(\pi_1(\mathcal{V}', \nabla'))^*$ respectively. Hence $\alpha \in \text{Hom}_{\text{Rep}(\pi_1(X, x))}(V_x, V'_x)$, where $V_x$ and $V'_x$ are fibers over $x$ of $\mathcal{V}$ and $\mathcal{V}'$ respectively. The fact that the functor $G$ is an equivalence of categories implies that $\alpha = F(f)$ for some $f : (\mathcal{V}, \nabla) \to (\mathcal{V}', \nabla')$.

Let $\tau : V \to \mathcal{H} \otimes V$ be a representation of $\mathcal{H}$. Then $\tau^* : (\text{Spec } \mathcal{H})(C) \to \text{Aut}(V)$ is a representation of a group $(\text{Spec } \mathcal{H})(C)$. The principal $(\text{Spec } \mathcal{H})(C)$-bundle $(\text{Spec } \mathcal{H}^0)(C)$ is equipped with the integrable connection, hence the associated vector bundle $\mathcal{V}(\tau) := (\text{Spec } \mathcal{H}^0)(C) \times_{\tau^*} V \to X$ is equipped with an integrable connection $\nabla_{\mathcal{V}(\tau)}$ induced by the connection of the principal bundle. The monodromy representation of $\nabla_{\mathcal{V}(\tau)}$ is equal $\tau^* \circ \Theta_x$. On the other hand it is equal $(\pi_{\nabla_{\mathcal{V}(\tau)}})^* \circ \Theta_x$.

6.1. Now we shall prove an algebraic analogue of Theorem 6.0.1.

Let $\tilde{X}$ be a smooth proper scheme of finite type over $\text{Spec } k$ and let $D$ be a divisor with normal crossings in $\tilde{X}$. Let $X := \tilde{X} \setminus D$.

We assume that $P \to X$ is an algebraic principal $G$-bundle equipped with an algebraic integrable regular connection. Let $\sigma : k \hookrightarrow C$ be an embedding. Let us set $X_{\sigma} := X \times_k C$. We assume that the image of the monodromy homomorphism at $x$

$$\theta : \pi_1(X_{\sigma}(C), x) \to G(C)$$

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is Zariski dense in $G$.

Let $DE(X, \theta)_{alg}$ be a category of algebraic vector bundles $\mathcal{V}$ on $X$ equipped with an increasing filtration $\{W_i\mathcal{V}\}$ by algebraic subvector bundles and with an algebraic integrable regular connection $\nabla_\mathcal{V}$ compatible with the filtration $\{W_i\mathcal{V}\}$ such that the pair $(Gr_W \mathcal{V}, \nabla_{Gr_W \mathcal{V}})$ is of the form $(P \times_\rho V; \nabla_\rho)$.

We recall from previous sections that given an algebraic principal $G$-bundle equipped with an algebraic integrable regular connection we have constructed a Hopf $k$-algebra $\mathcal{H}$ and a bundle $\mathcal{H}^0$ equipped with a regular integrable connection.

**Theorem 6.1.1.** The categories $DE(X, \theta)_{alg}$ and $Rep(\mathcal{H})$ are equivalent.

**Proof.** First we consider the case $k = \mathbb{C}$. Let $h : DE(X, \theta)_{alg} \to DE(X, \theta)_{hol}$ be a functor which to an algebraic object associates a corresponding analytic object. This functor is an equivalence of categories. Hence the functor

$$F : DE(X, \theta)_{alg} \to Rep(\mathcal{H})$$

is an equivalence of categories. This implies that $F$ is an equivalence of categories for any algebraically closed field $k$ contained in $\mathbb{C}$. Let $k \subset \mathbb{C}$ and let $\bar{k} \subset \mathbb{C}$ be its algebraic closure. We have a commutative diagram of functors

$$\begin{array}{ccc}
DE(X\theta)_{alg} & \xrightarrow{F_k} & Rep(\mathcal{H}) \\
i_1 & & \downarrow i_2 \\
DE(X \times_k \bar{k}, \theta)_{alg} & \xrightarrow{F_k} & Rep(\mathcal{H} \otimes_k \bar{k}).
\end{array}$$

We shall use the following observation. If a system of linear equations over $k$ has a generic solution in $\bar{k}$, then it has a generic solution in $k$. This implies that $i_1$ and $i_2$ are injective on classes of isomorphisms of objects. The functors $i_1$ and $i_2$ on morphisms behaves like tensoring with $\bar{k}$ over $k$. Hence the functor $F_k$ is an equivalence of categories.

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The fiber of $Hi(trp)$ over $(y,X \times y)$ times $X$ can be describe in the following way:

We choose a point $X$ prime in $\pi-1(X)$. Hence we get an isomorphism $P \approx G$. The restriction of $Ome y$ times $X$ n+1 (p ot (n+1)) to $y$ times $X$ times $X$ is ot og. Hence the fiber of $Hi(trp)$ over $(y,X)$ is $Hi \ (t \ Rga \ Ome \ (X \ y,x)(\text{Del 1} ;y,X) \ (p \ ot \ bullet) \ ot \ og.$

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A. Cosimplicial spaces.

A.1. We review here some definitions from [W]. The category $\Delta$ is defined in the following way. The objects of $\Delta$ are sequences of integers $\Delta_n = (0, 1, \ldots, n)$. The morphisms of $\Delta$ are monotonic maps $\mu : \Delta_n \to \Delta_m$. Morphisms $\delta^i : \Delta_{n-1} \to \Delta_n$ for $n \geq i \geq 0$ given by $\delta^i(j) = j$ if $j < i$ and $\delta^i(j) = j + 1$ if $j \geq i$ are called coface operators. Morphisms $s^j : \Delta_n \to \Delta_{n-1}$ for $n - 1 \geq j \geq 0$ given by $s^j(k) = k$ if $j \geq k$ and $s^j(k) = k - 1$ if $k \geq j + 1$ are called codegeneracy operators. A cosimplicial object in a category $C$ is a covariant functor $X : \Delta \to C$. We usually denote $X(\Delta_n)$ by $X^n$, $X(\delta^i)$ by $\delta^i$ and $X(s^j)$ by $s^j$.

A.1. Let $K^{*,\bullet}$ be a bicomplex with commuting differentials $\partial^{i,j} : K^{i,j} \to K^{i+1,j}$ and $\delta^{i,j} : K^{i,j} \to K^{i,j-1}$. We define the total complex of $K^{*,\bullet}$ in the following way:

$$(\text{Tot}K^{*,\bullet})_m := \bigoplus_{i-j=m} K^{i,j},$$

$$d_m : (\text{Tot}K^{*,\bullet})_m \to (\text{Tot}K^{*,\bullet})_{m-1}$$

and

$$d_m|_{K^{i,j}} = \partial^{i,j} + (-1)^i \delta^{i,j}.$$ 

Let $X^\bullet$ be a cosimplicial space. A sheaf on $X^\bullet$ consists of sheaves $F_n$ on $X^n$ together with maps $F_m \to \alpha_* F_n$ for any $\Delta_n \to \Delta_m$ in $\Delta$ satisfying the obvious compatibility conditions.

If $F_\bullet$ is a sheaf on $X^\bullet$ with values in an abelian category $C$ then the global section functor on $X^\bullet$,

$$\Gamma(F_\bullet; X^\bullet) : n \to \Gamma(F_n; X^n)$$

is a simplicial object in $C$. The obvious functor

$$(\text{simplicial objects in } C ) \to (\text{complexes in } C )$$

associates to $\Gamma(F_\bullet; X^\bullet)$ and hence also to $F_\bullet$, a complex which we shall denote also by $\Gamma(F_\bullet; X^\bullet)$.

A.2. Let $X^\bullet$ be a cosimplicial space. A category of sheaves of abelian groups on $X^\bullet$ is an abelian category which we denote by $\text{Ab}(X^\bullet)$. If $F_\bullet$ is a sheaf of abelian groups on $X^\bullet$, then $F_\bullet$ has a right resolution $K^\bullet_\bullet$ in $\text{Ab}(X^\bullet)$ such that $H^r(X^q; K^p_q) = 0$ for $r > 0$. One can take a canonical Godement resolution. The resolution $K^\bullet_\bullet$, after applying the functor of global sections leads to a bicomplex $\Gamma(K^\bullet_\bullet; X^\bullet)$. One defines

$$H^n(X^\bullet; F_\bullet) := H^n(\text{Tot}\Gamma(K^\bullet_\bullet; X^\bullet)).$$

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Let $F^*_n$ be a complex of sheaves of abelian groups on $X^\bullet$. One shows that there is a quasi-isomorphism $F^*_n \to K^*_n$ such that $H^r(X^q; K^*_n) = 0$ for $r > 0$. One can take for each pair $(n, m)$ a canonical Godement resolution of $F^m_n$, $C^*(X^n; F^m_n)$. Then $C^p(X^n; F^q_n)_{p,q}$ is a bicomplex of sheaves on $X^n$ and $K^*_n$ is its total complex. We define (hyper-) cohomology of $X^\bullet$ with coefficients in $F^*_n$ in the following way:

$$H^n(X^\bullet; F^*_n) := H^n(\text{Tot}(K^*_n; X^\bullet)).$$

A.3. Let $A^*_n$ be a complex of sheaves of $k$-vector spaces on a cosimplicial space $X^\bullet$. We say that $A^*_n$ is a sheaf of commutative differential graded $k$-algebras on $X^\bullet$ if for each $n$, $A^*_n$ is a sheaf of commutative differential graded $k$-algebras on $X^n$ and if the structure maps are morphisms of $k$-algebras.

**Example.** $\Omega^*_n$ (in degree $n$, on $X^n$, we have the De Rham complex $\Omega^*_n$) is a sheaf of commutative differential graded $k$-algebras on $X^\bullet$.

It follows from [N] that for any sheaf $A^*_n$ of cdg $k$-algebras on $X^\bullet$, there is a sheaf $I^*_n$ of cdg $k$-algebras on $X^\bullet$, and a quasi-isomorphism, a morphism of $k$-algebras $A^*_n \to I^*_n$ such that $I^*_n$ is a complex of flasque sheaves for each $n$. Then $(n \to \Gamma(I^*_n, X^n))$ is a simplicial object in the category of commutative graded differential $k$-algebras. We shall describe how to introduce the product in the cohomology of the total complex.

Let $A^{*,*} = \{n \to A^{*,n}\}_n$ be a simplicial object in the category of cdg $k$-algebras with face maps $\delta_i$ and degeneracy maps $s_j$. We shall define a shuffle product in the total complex $\text{Tot}A$ by the following formula:

$$x * y := \sum_{(q_1, q_2) - \text{shuffles } \sigma^{-1}} (s_{\sigma(q_2+q_1)} \circ \ldots \circ s_{\sigma(q_1+1)}(x)) \cdot (s_{\sigma(q_1)} \circ \ldots \circ s_{\sigma(1)})(y),$$

$$\pi_1(X, x) \xrightarrow{\theta} \text{Spec } \mathcal{H} \xrightarrow{\tau^*} \text{End}(V).$$

**References**


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